Hierarchical Tucker Tensor Optimization - Applications to 4D Seismic Data Interpolation
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Motivation

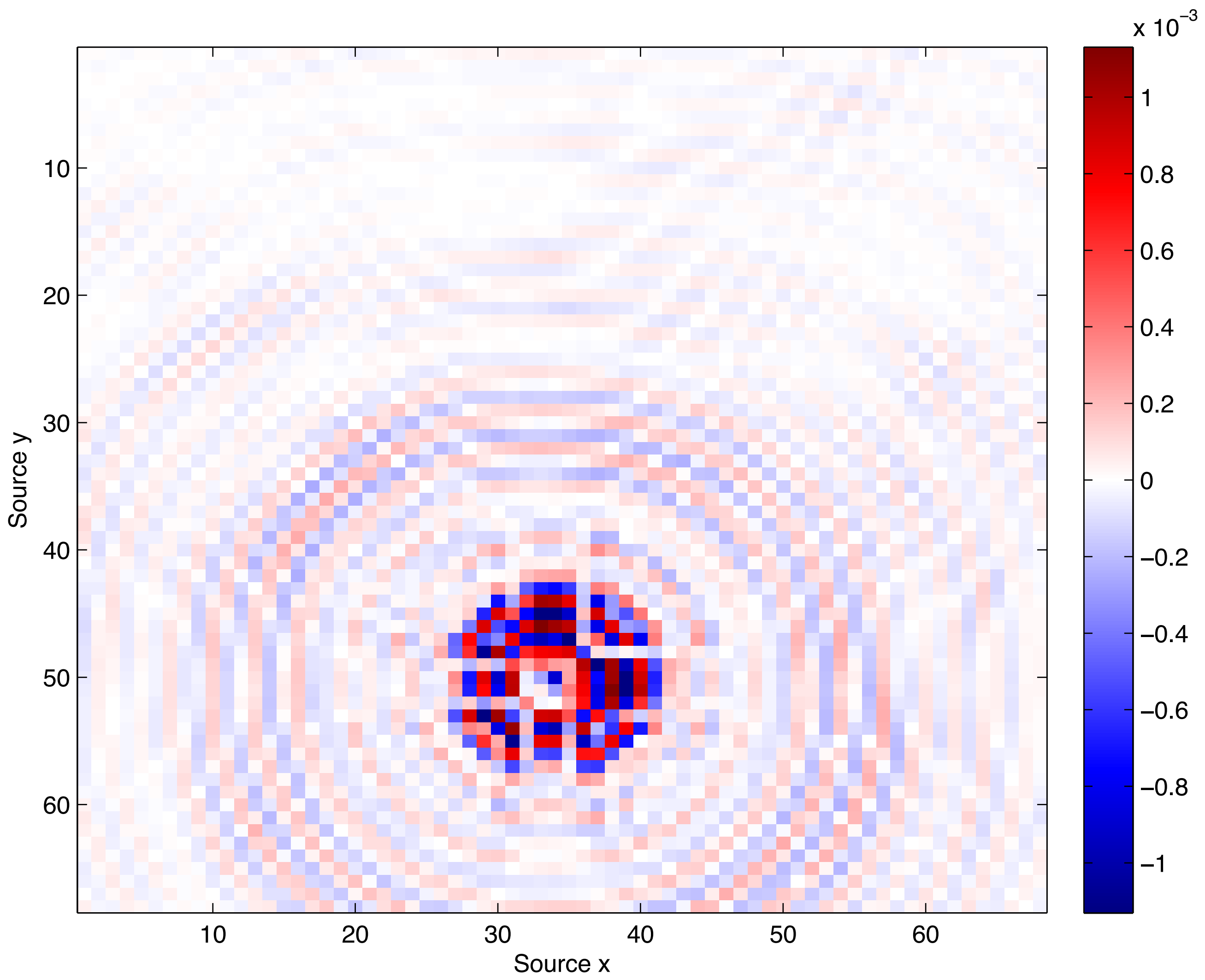
- 3D seismic experiments - 5D data
  - expensive to acquire, store
- Structured data - interpolation
- Fully sampled data
  - simultaneous sources in wave-equation based inversion
  - simulating multiples

Goals

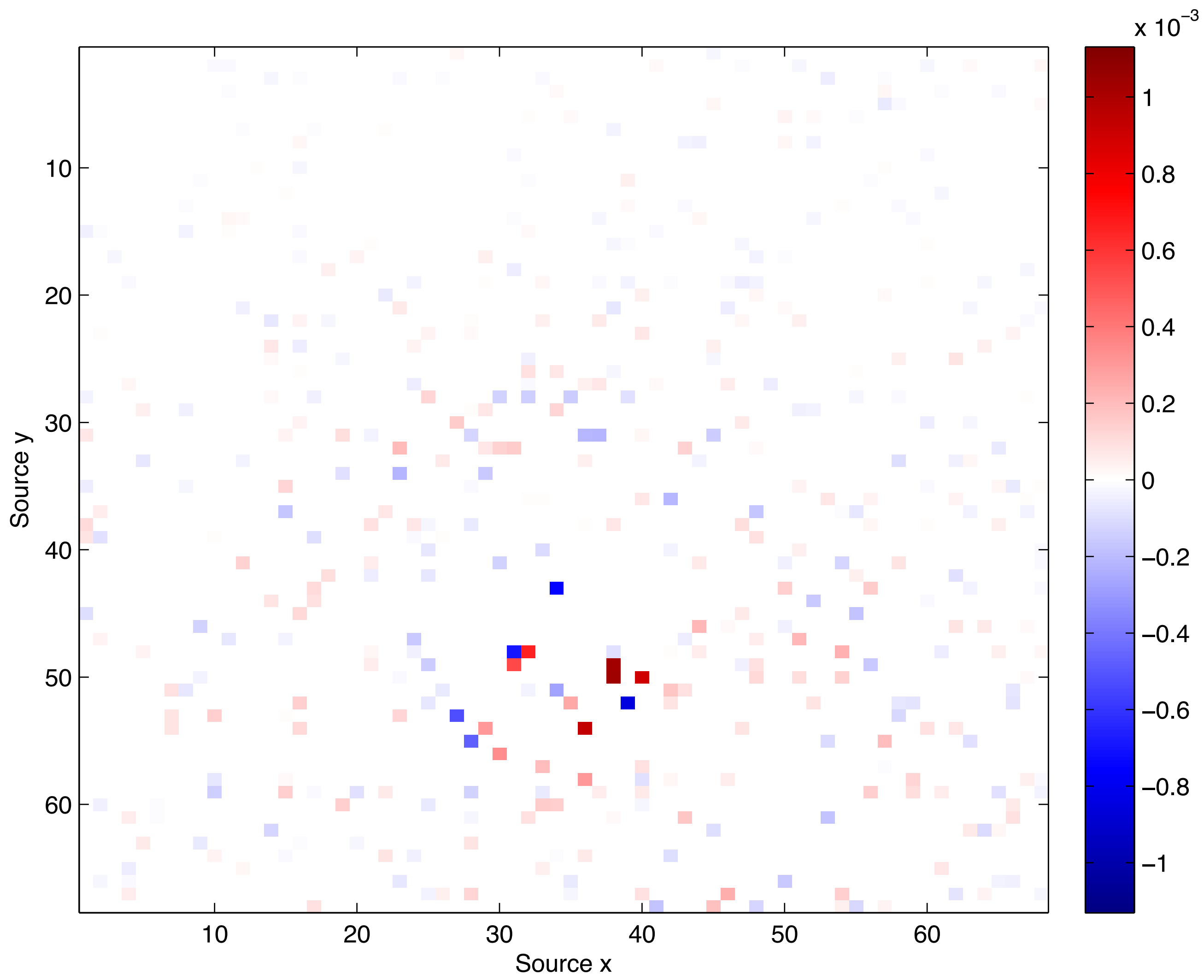
- Generalization of Compressive Sensing to multiple dimensions
- Randomized source/receiver acquisition
  - Reduce acquisition financial/time costs

7.34 Hz - 75% missing sources

Common receiver gather



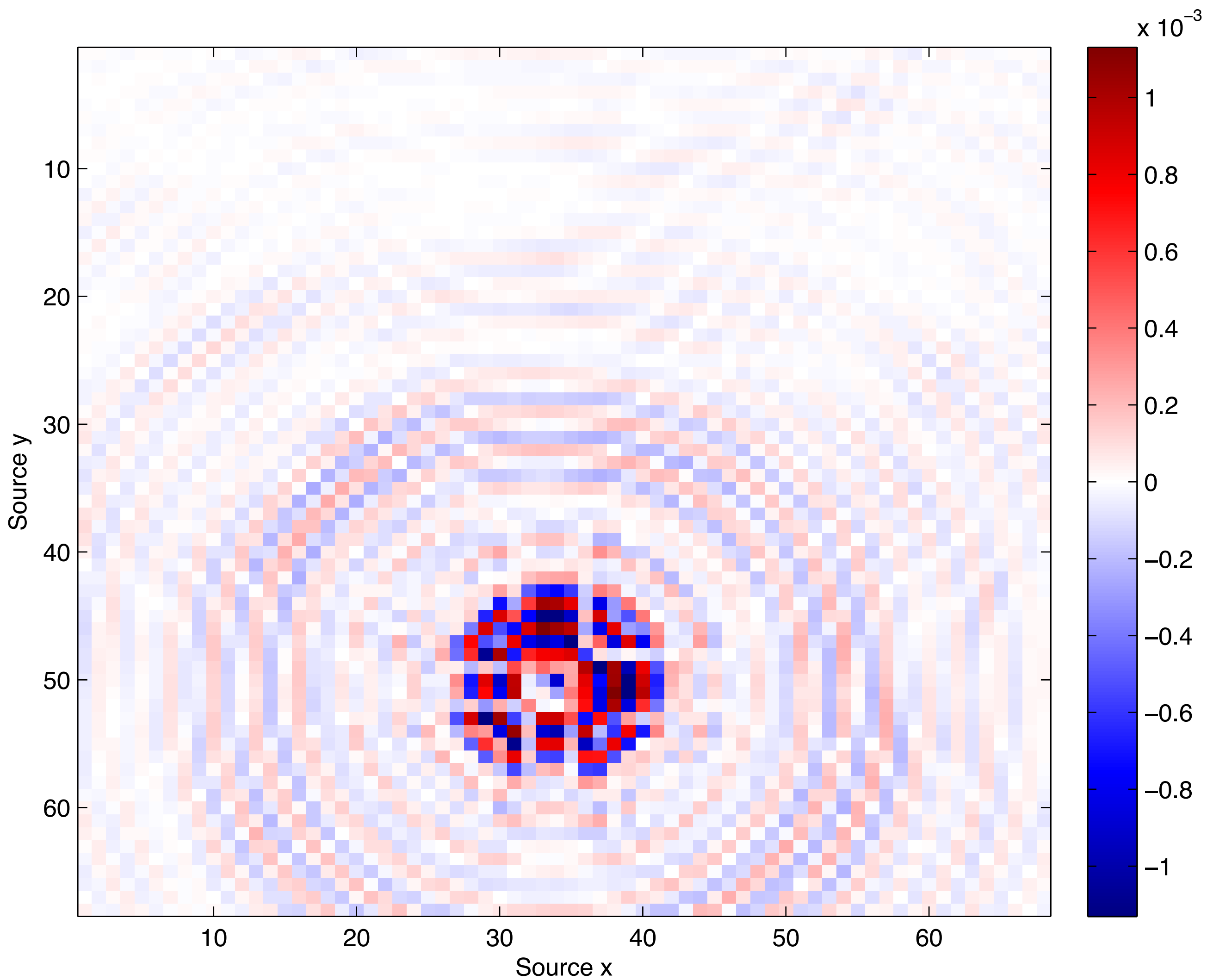
$(x_{rec}, y_{rec}) = (75,50)$

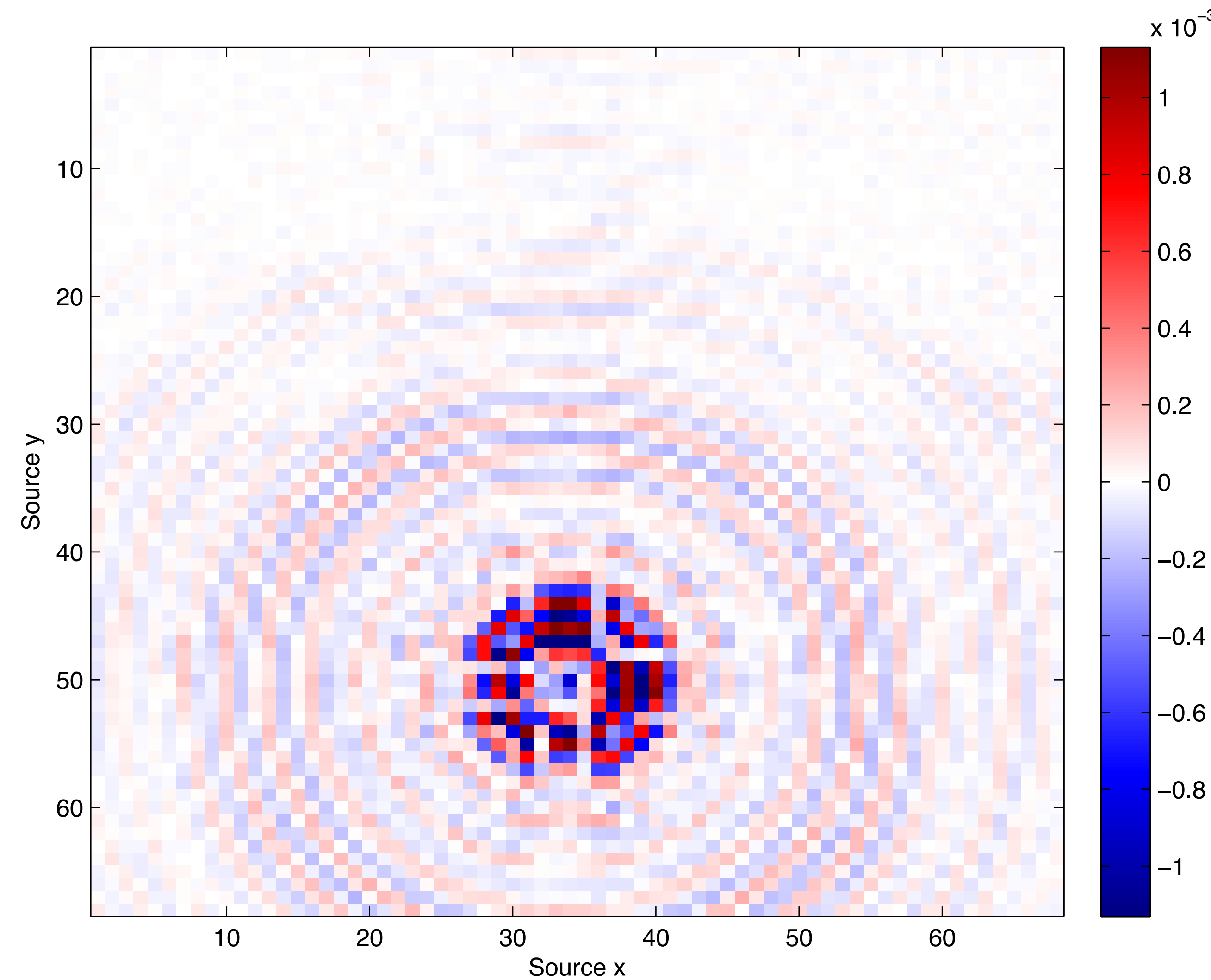


Subsampled Data

7.34 Hz - 75% missing sources

Common receiver gather





$(x_{rec}, y_{rec}) = (75,50)$

Interpolated Data

SNR 11.3 dB

Compressive sensing

with sparsity promotion

Successful reconstruction scheme

- Signal structure - *sparsity*
- Sampling - *subsampling decreases sparsity*
- Optimization - *look for sparsest solution*

Multidimensional interpolation

with Hierarchical Tucker

Successful reconstruction scheme

- **Signal structure - *Hierarchical Tucker***
- Sampling - *subsampling increases h-rank*
- Optimization - *fit data in the Hierarchical Tucker format*

Matricization

- The *matricization* of a tensor X with dimensions $1, \ldots, d$ along the dimensions $t = (t_1, \ldots, t_r)$ is the matrix formed by placing the dimensions t along the rows and dimensions t^c along the columns
- Denoted $X^{(t)}$

Example in Matlab

```
n1 = 20; n2 = 20; n3 = 20; n4 = 20;
% Tensor
x = randn(n1,n2,n3,n4);
% Matricization along dimensions 1 and 2
x12 = reshape(x,n1 * n2, n3 * n4);
% Matricization along dimensions 3 and 4
y34 = permute(x,[3 4 1 2]);
x34 = reshape(x, n3 * n4, n1 * n2);
% Matricization along dimensions 1 and 3
y13 = permute(x,[1 3 2 4]);
x13 = reshape(x,n1 * n3, n2 * n4);
```

$X^{(1,2)}$

$X^{(3,4)}$

$X^{(1,3)}$

Kronecker product

A, B Matrices

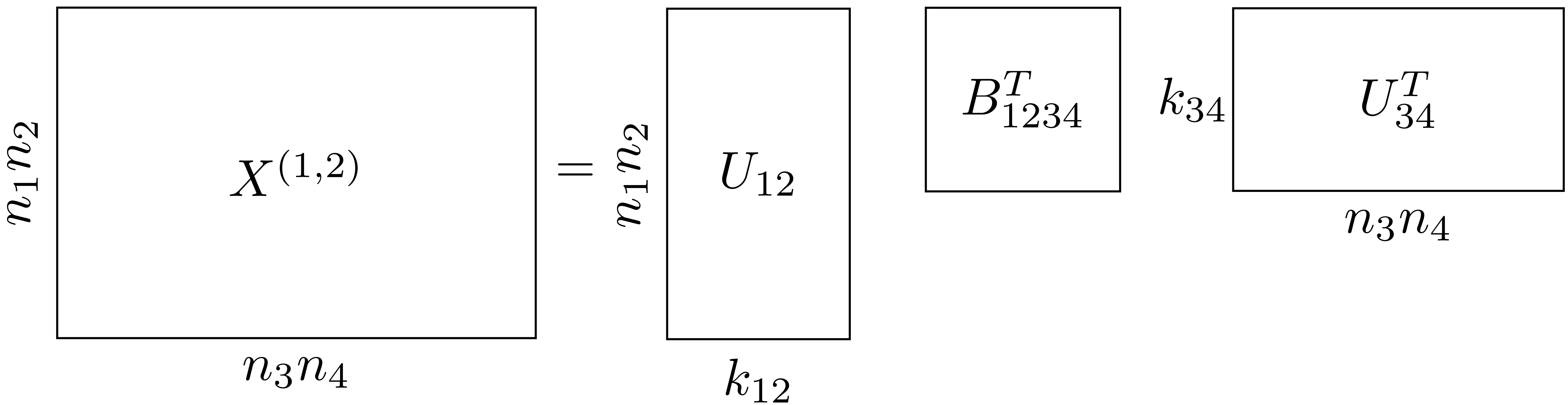
$B \otimes A$ Kronecker product

$$(B \otimes A)\text{vec}(X) = \text{vec}(AXB^T)$$

A acts along the columns of X

B acts along the rows of X

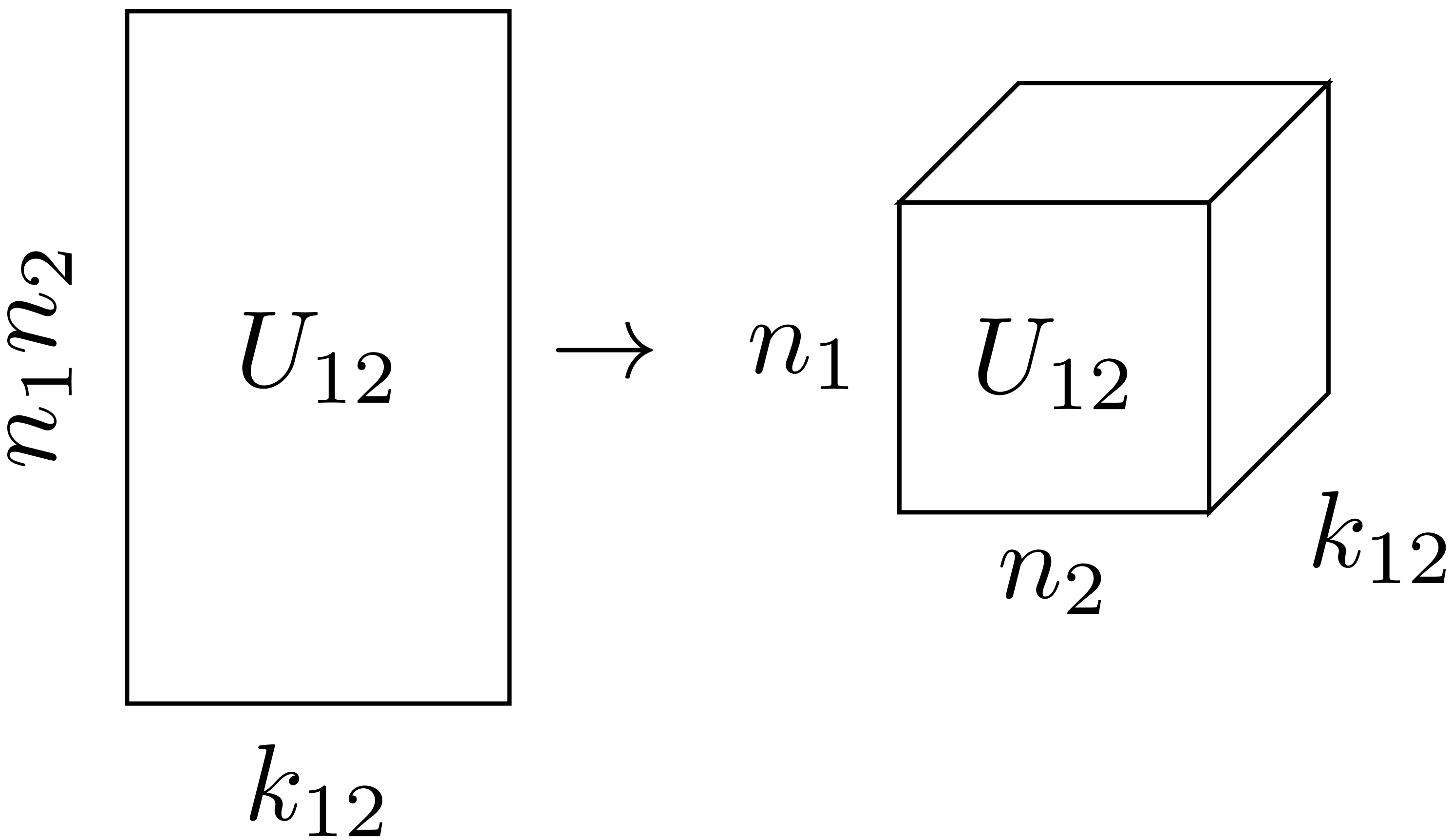
HT Format $X - n_1 \times n_2 \times n_3 \times n_4$ tensor



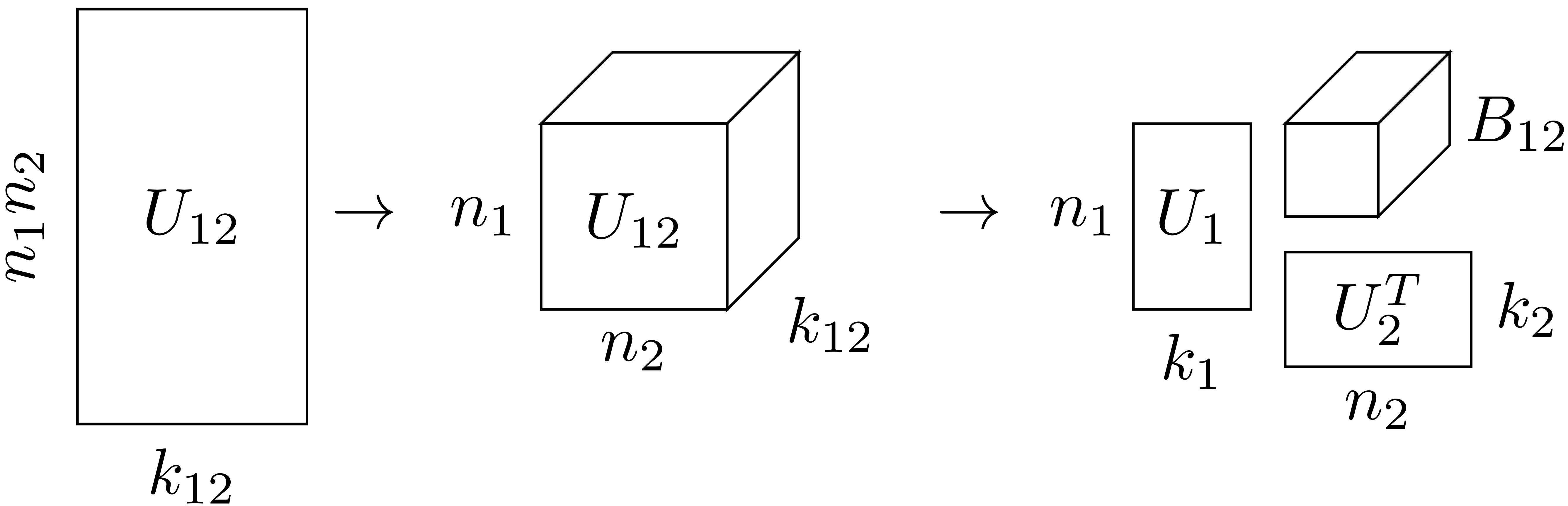
$$\mathrm{vec}(X) = (U_{12} \otimes U_{34})\mathrm{vec}(B_{1234})$$

$$X^{(1,2)} = U_{12} B_{1234}^T U_{34}^T$$

HT Format $X - n_1 \times n_2 \times n_3 \times n_4$ tensor



HT Format $X - n_1 \times n_2 \times n_3 \times n_4$ tensor

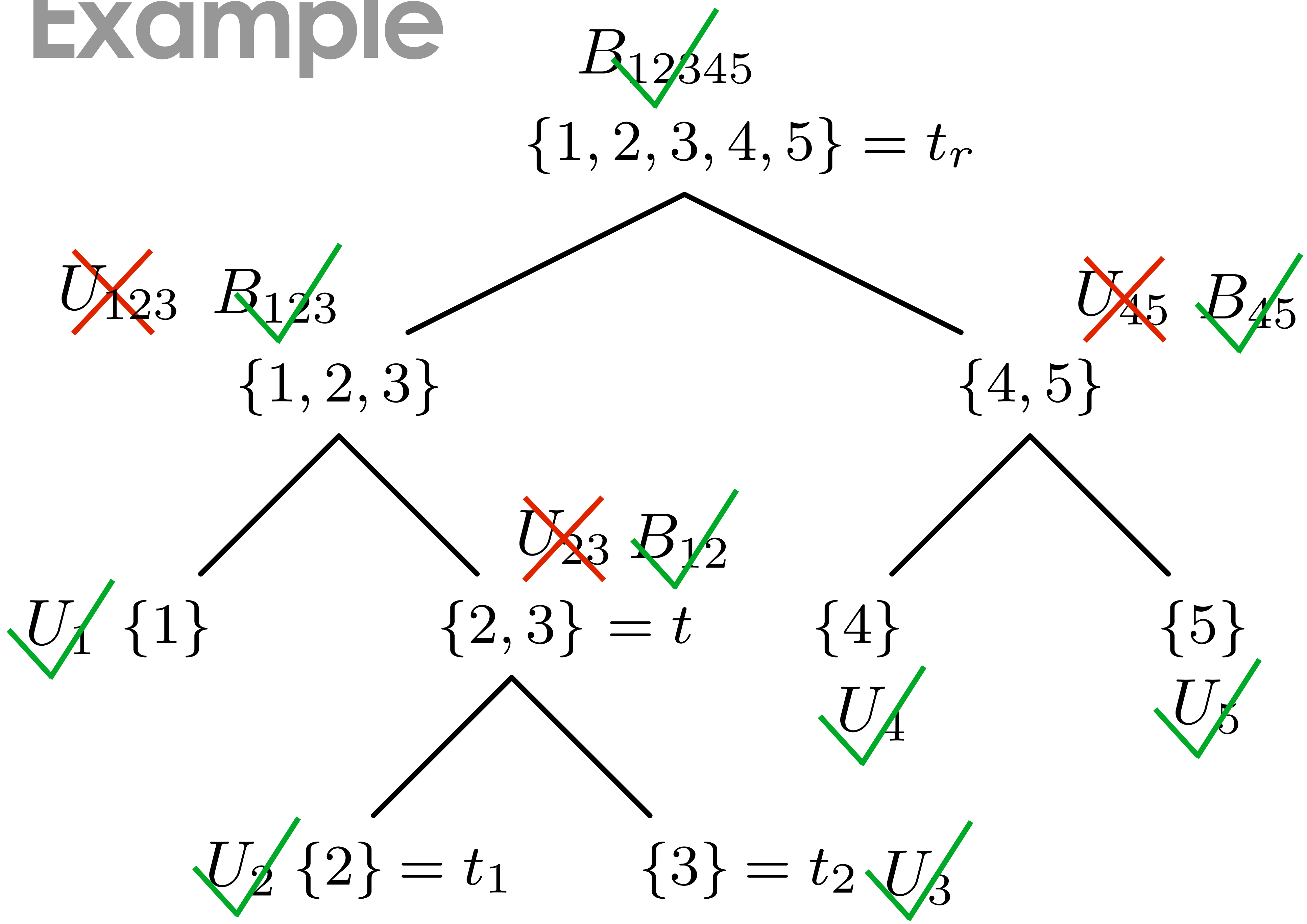


Hierarchical Tucker Format

- Intermediate matrices don’t need to be stored
- U_t, B_t - small parameter matrices
  - specify the tensor completely

A. Uschmajew, B. Vandereycken. The geometry of algorithms using hierarchical tensors. 2012

Example



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Hierarchical Tucker Format

- Storage $\leq dNK + (d-2)K^3 + K^2$
- Compare to N^d storage for the full tensor
- Effectively breaking the curse of dimensionality when $K \ll N$
- Low frequency data compresses

Hierarchical Tucker Format

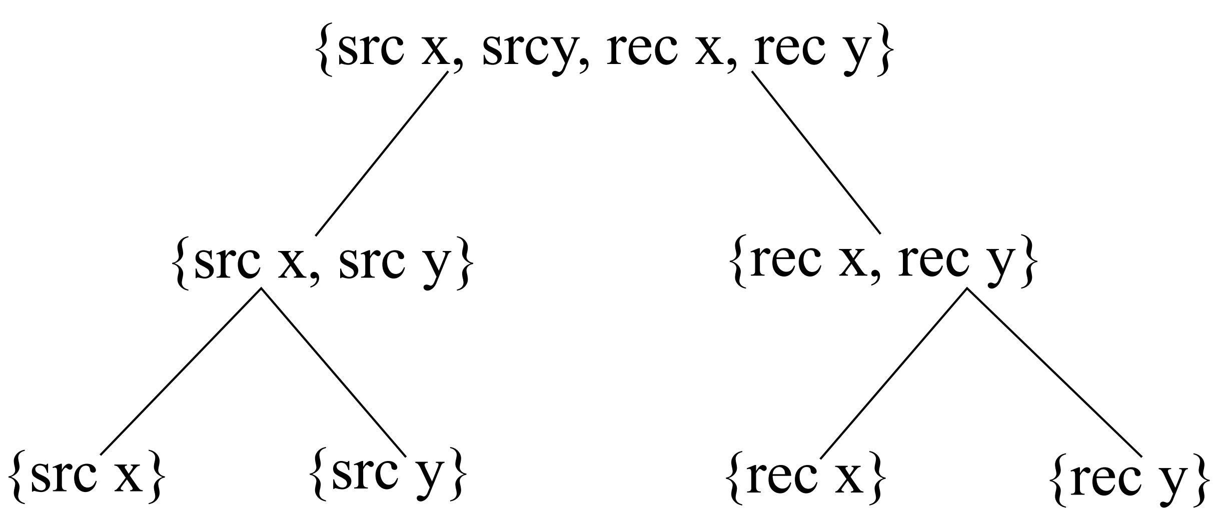
- E.g. for a 100 x 100 x 100 x 100 cube with max rank 20, $N = 100 \quad d = 4 \quad k = 20$
  Naive storage: $N^d = 10^8$ parameters
  HTucker storage: $= 24400$ parameters
  Compression of a factor of *99.97%*

Seismic HTucker

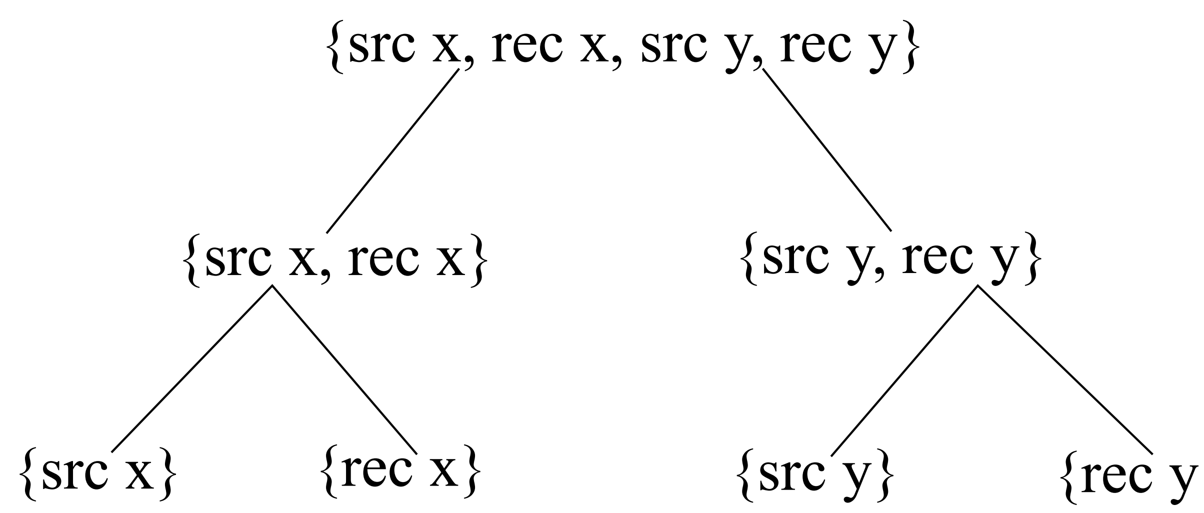
- We consider a 3D seismic survey with coordinates (src x, src y, rec x, rec y, time)
- We take a Fourier transform in time and restrict ourselves to a single frequency slice

Seismic HTucker

For a frequency slice with coordinates (src x, src y, rec x, rec y), there are essentially two choices of dimension splitting (by reciprocity)

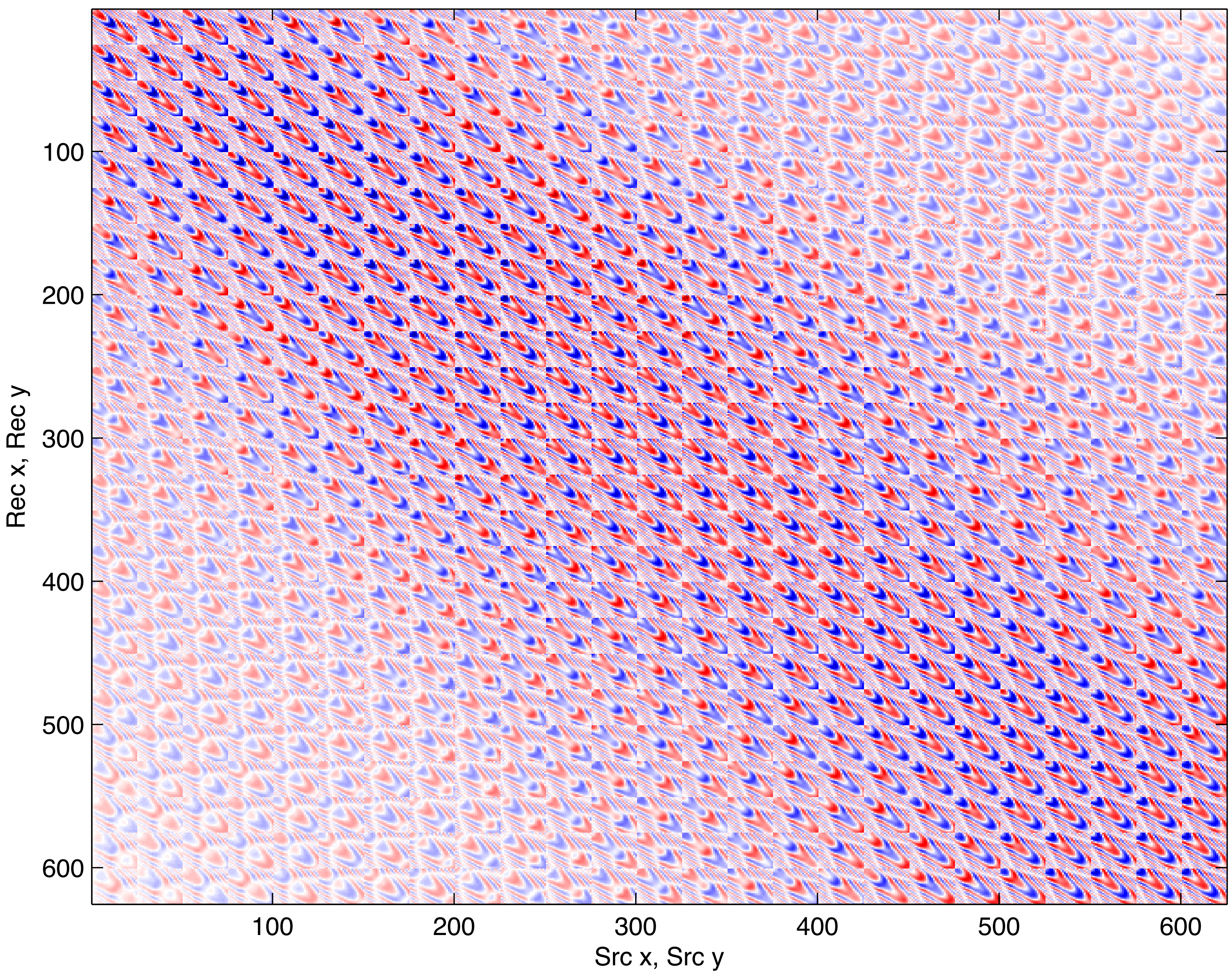


Canonical Decomposition

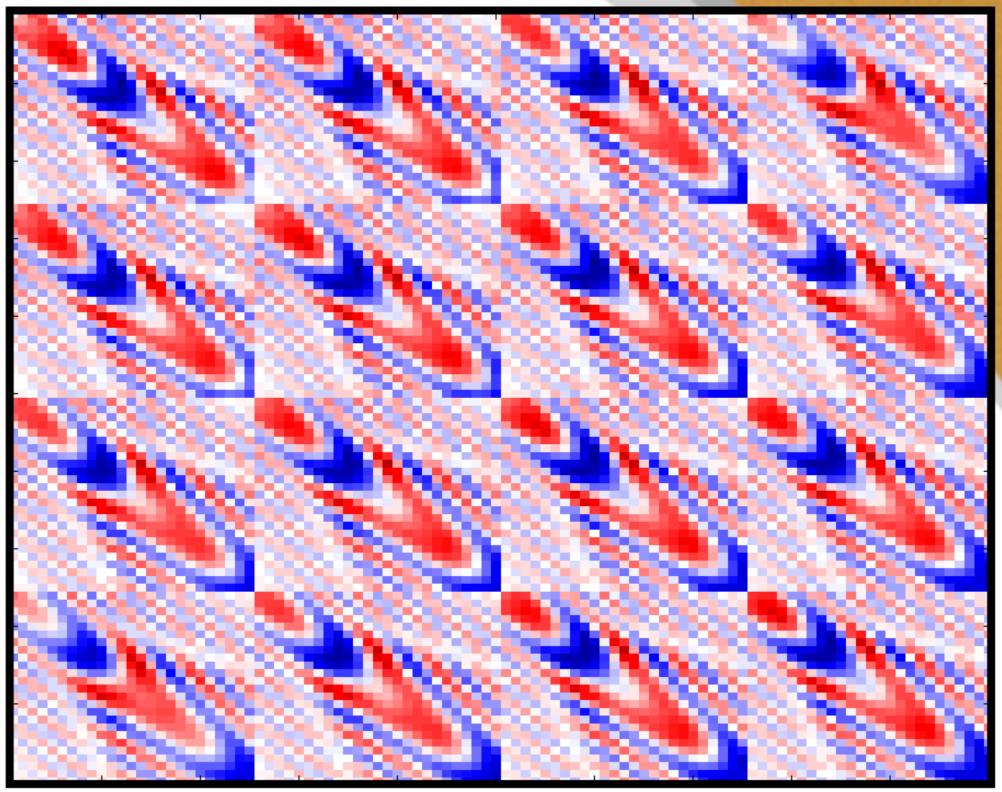


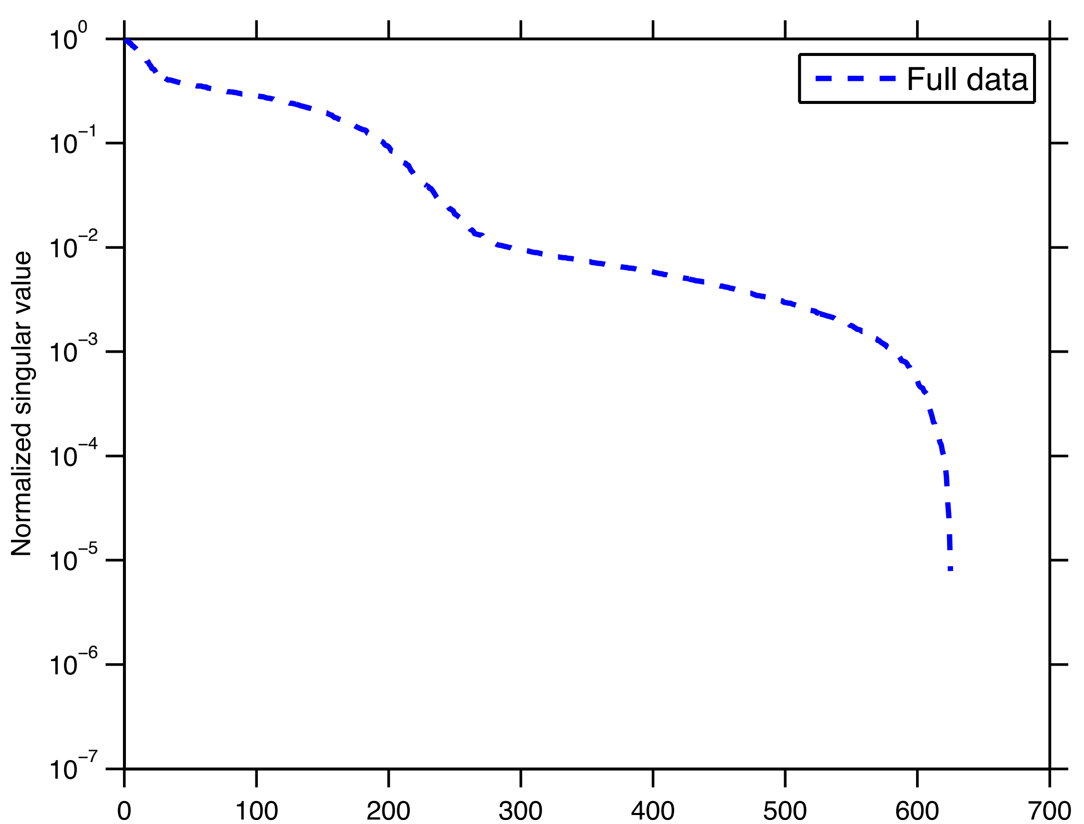
Non-canonical Decomposition

Matricizations

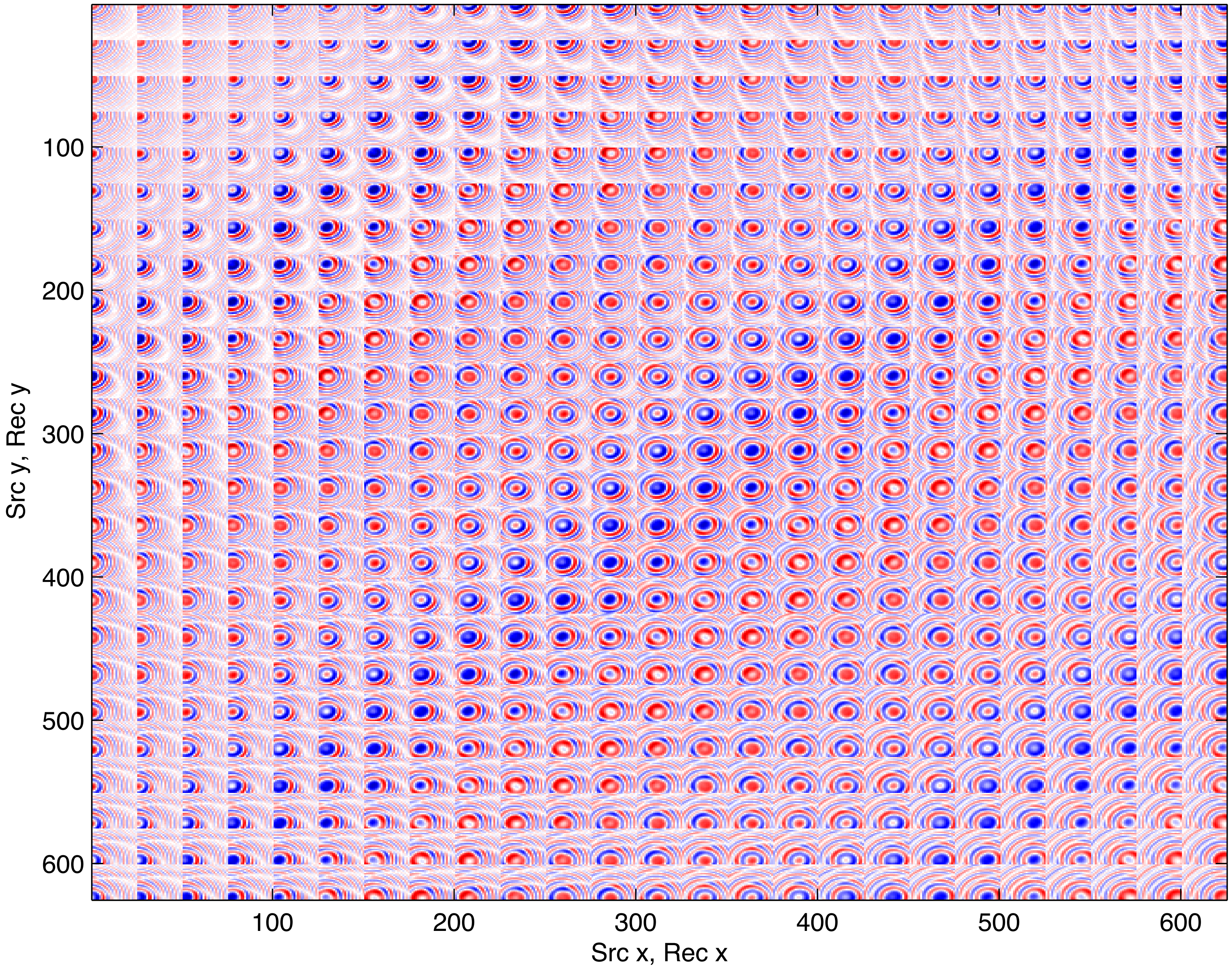


(Rec x, Rec y) matricization - full data

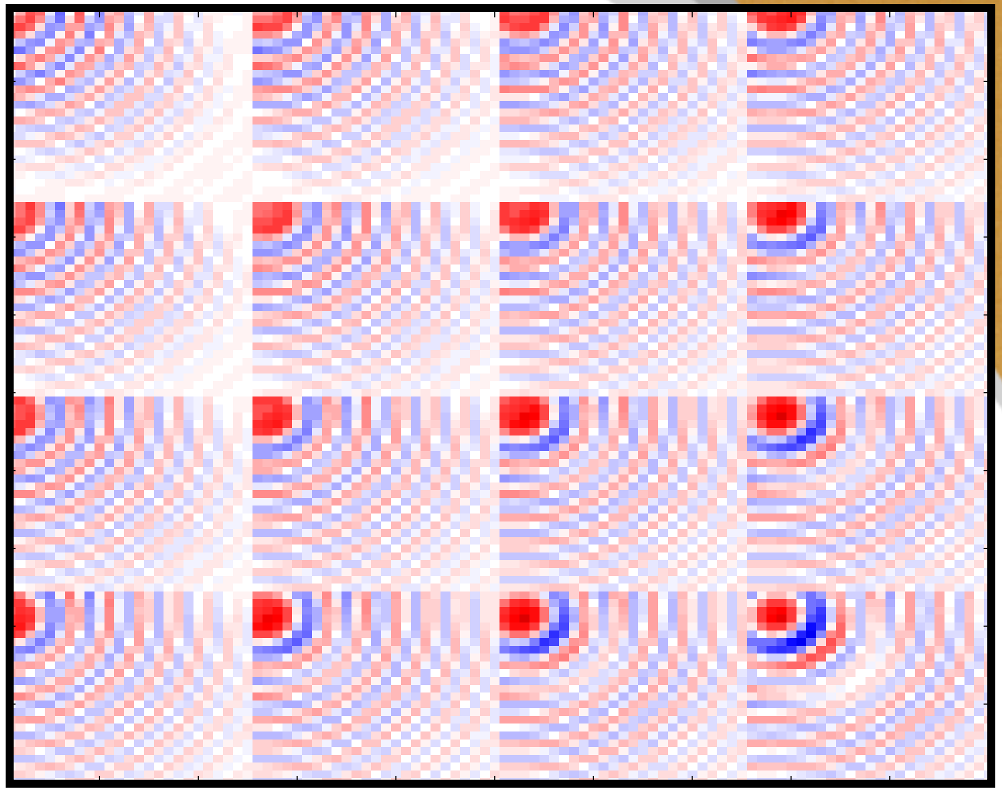


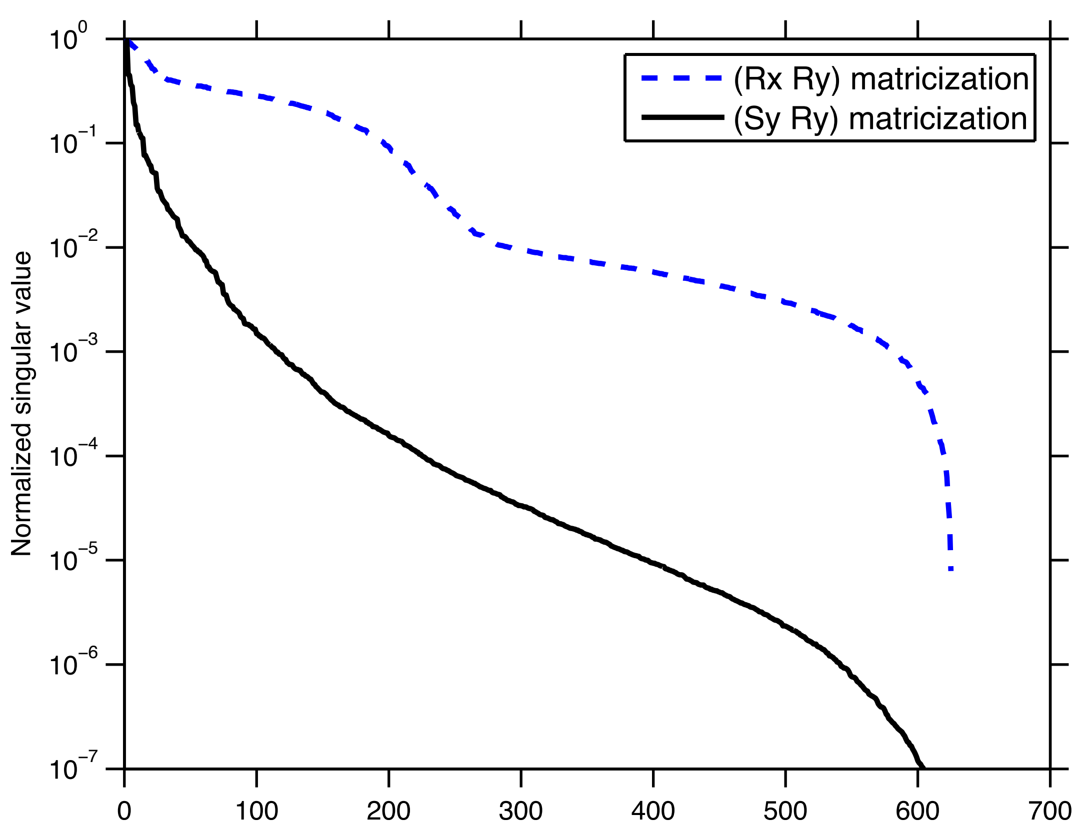


Matricizations



(Src y, Rec y) matricization - full data





Multidimensional interpolation

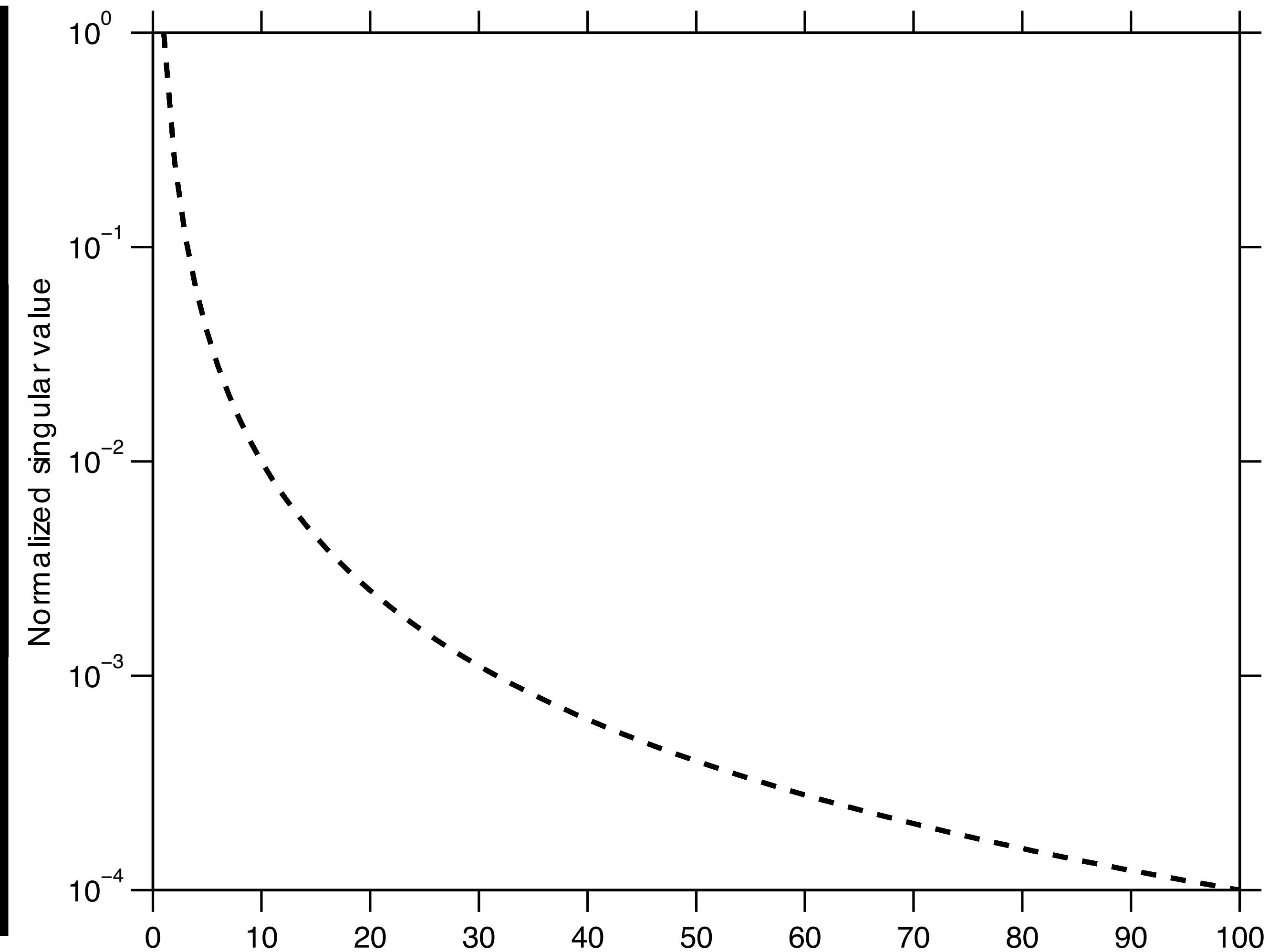
Successful reconstruction scheme

- Signal structure - *Hierarchical Tucker*
- **Sampling - *subsampling increases h-rank***
- Optimization - *fit data in the Hierarchical Tucker format*

Matrix Completion

$$\mathbf{X}$$

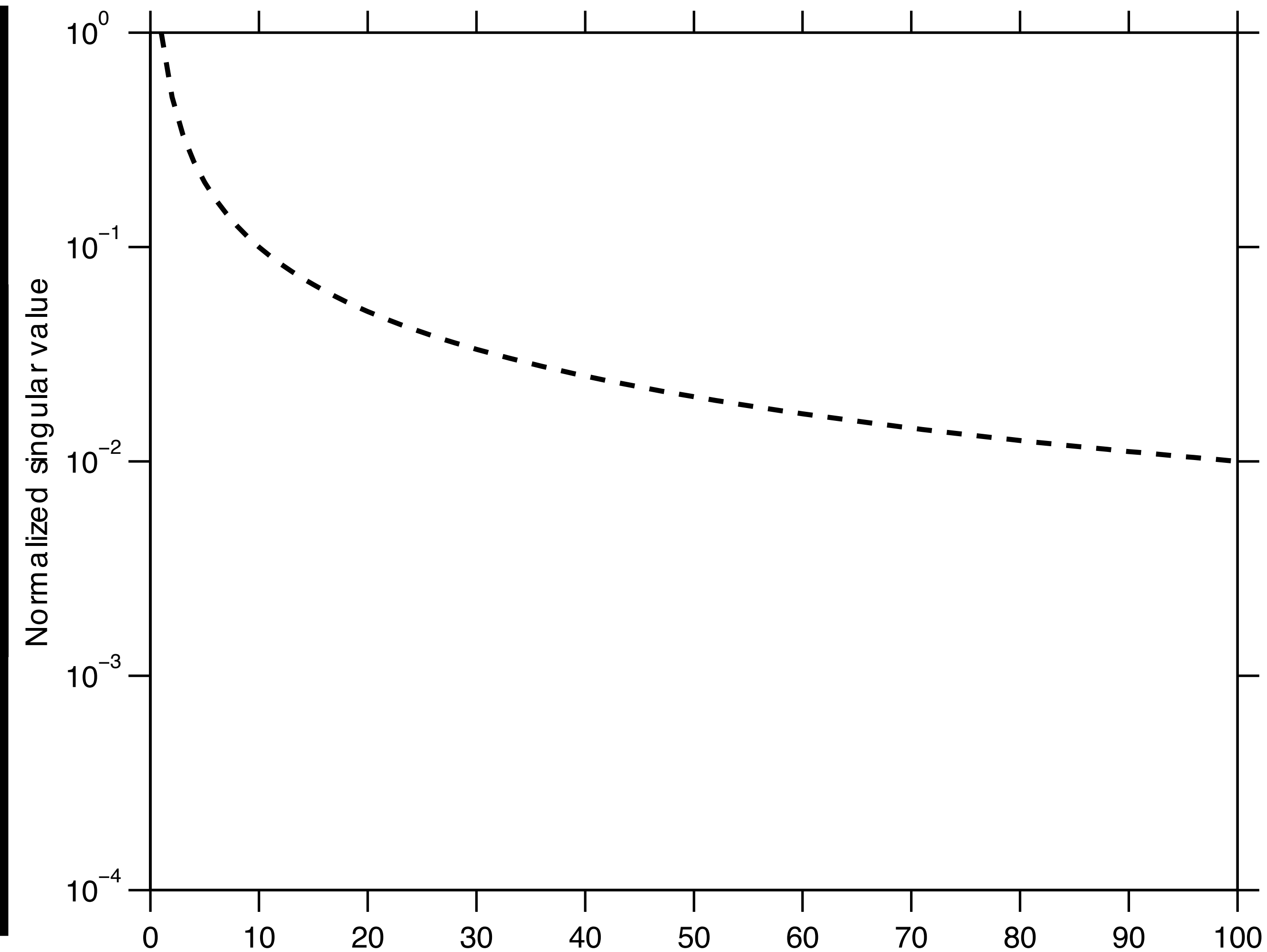
$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$



Matrix Completion

$$\mathcal{A}(\mathbf{X})$$

$$\begin{bmatrix} * & * & * & 0 & * \\ * & 0 & 0 & * & 0 \\ * & * & * & * & * \\ * & * & 0 & * & * \\ 0 & * & * & * & 0 \end{bmatrix}$$



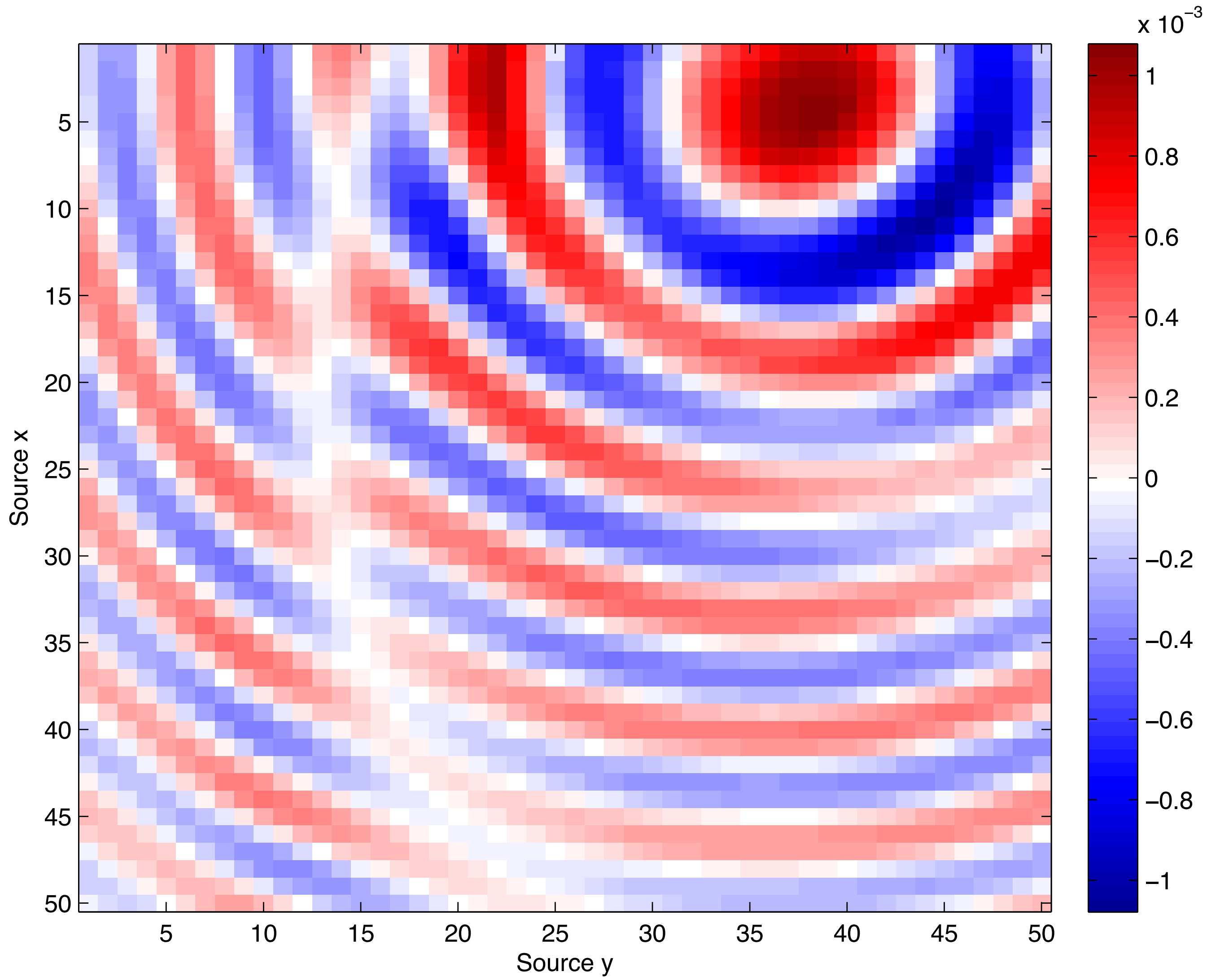
Tensor Completion

- *Structure* - recover a *tensor* $\mathbf{X}$ which has low *hierarchical* rank
  - Well represented in HT
- *Sampling* - random removal of points *increases* rank
  - Poorly represented in HT
  - Idealized sampling

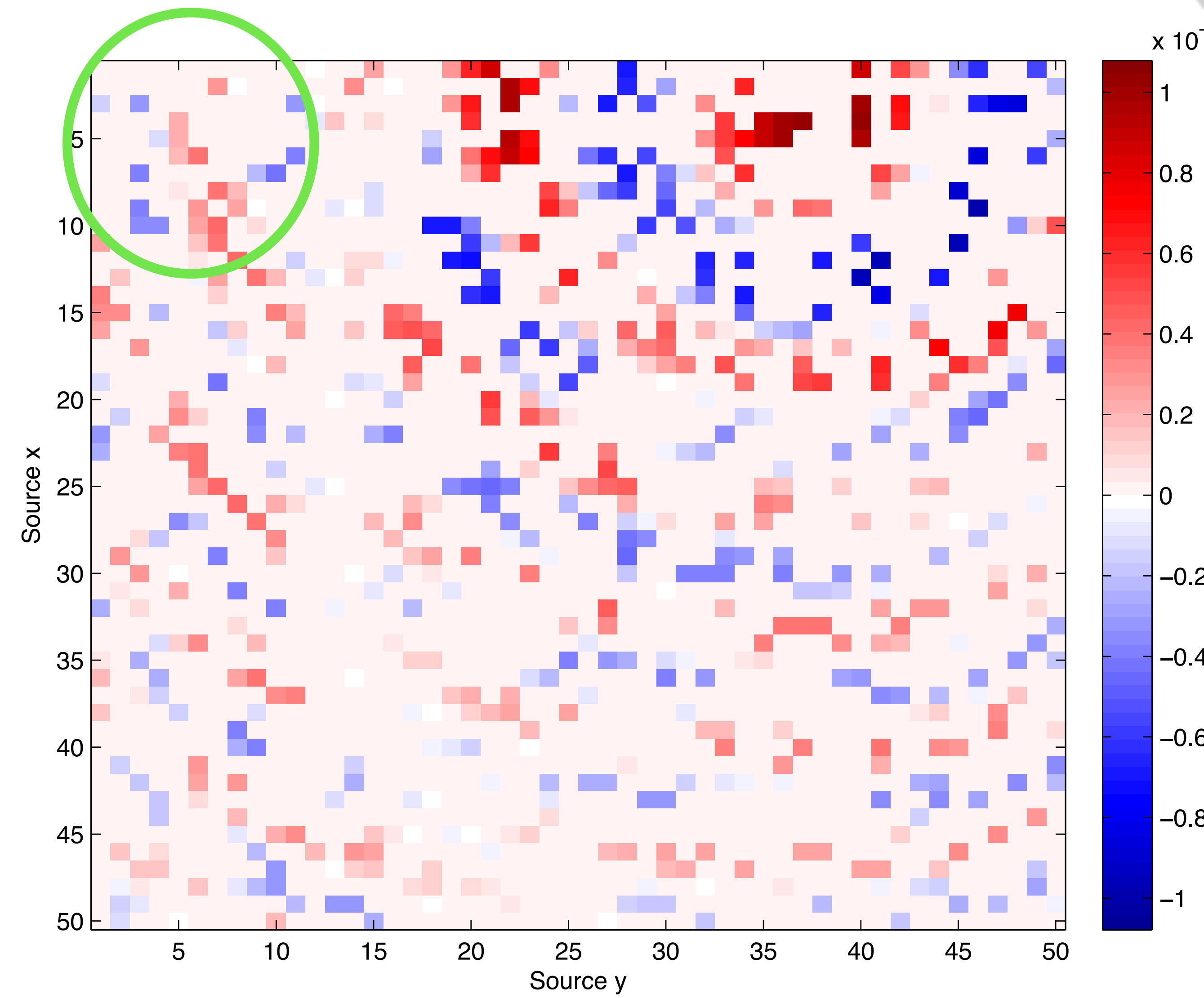
Idealized recovery

75% random entries removed

Common shot gather



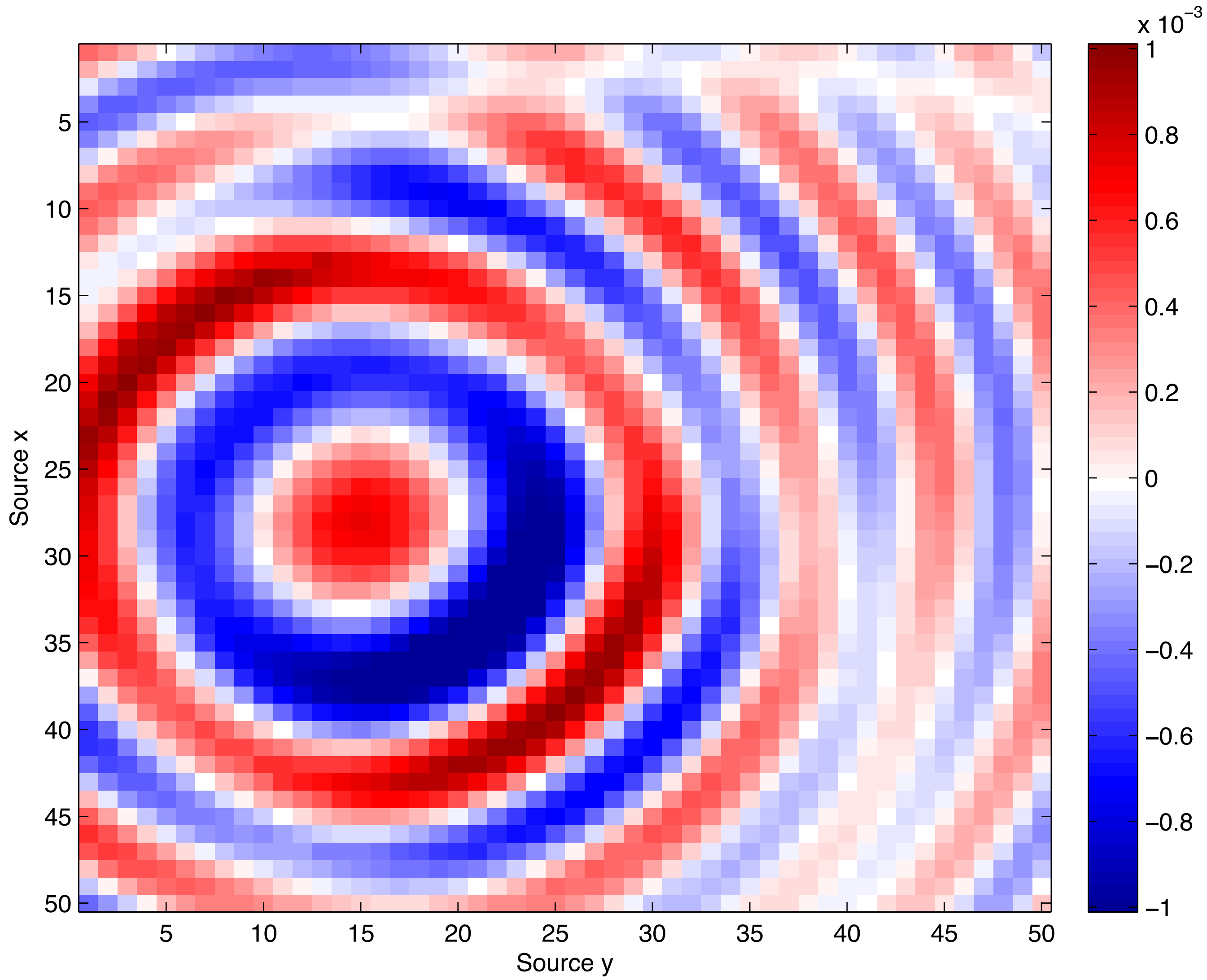
True Data



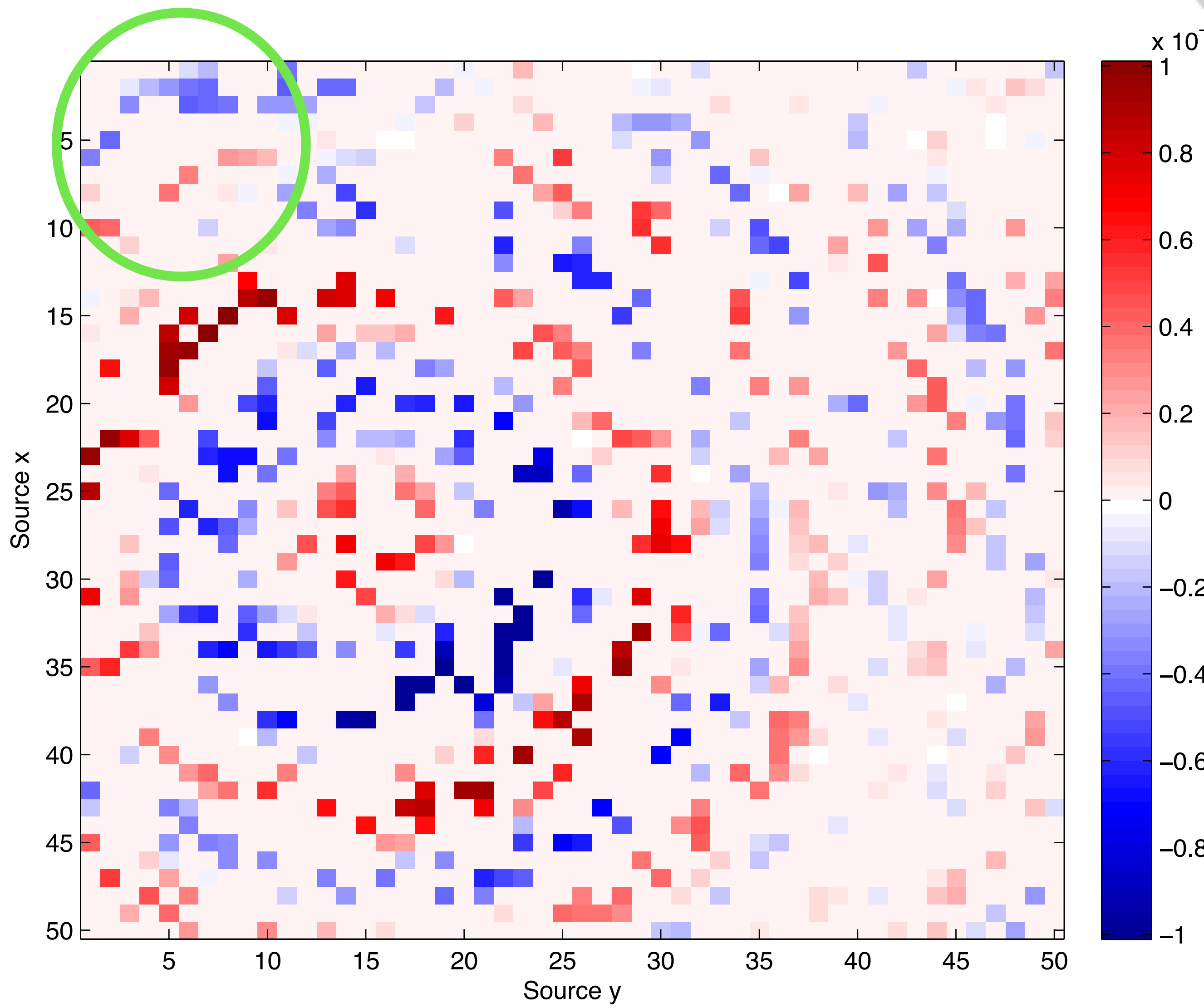
Subsampled Data

Idealized recovery

75% random entries removed
Common shot gather



True Data

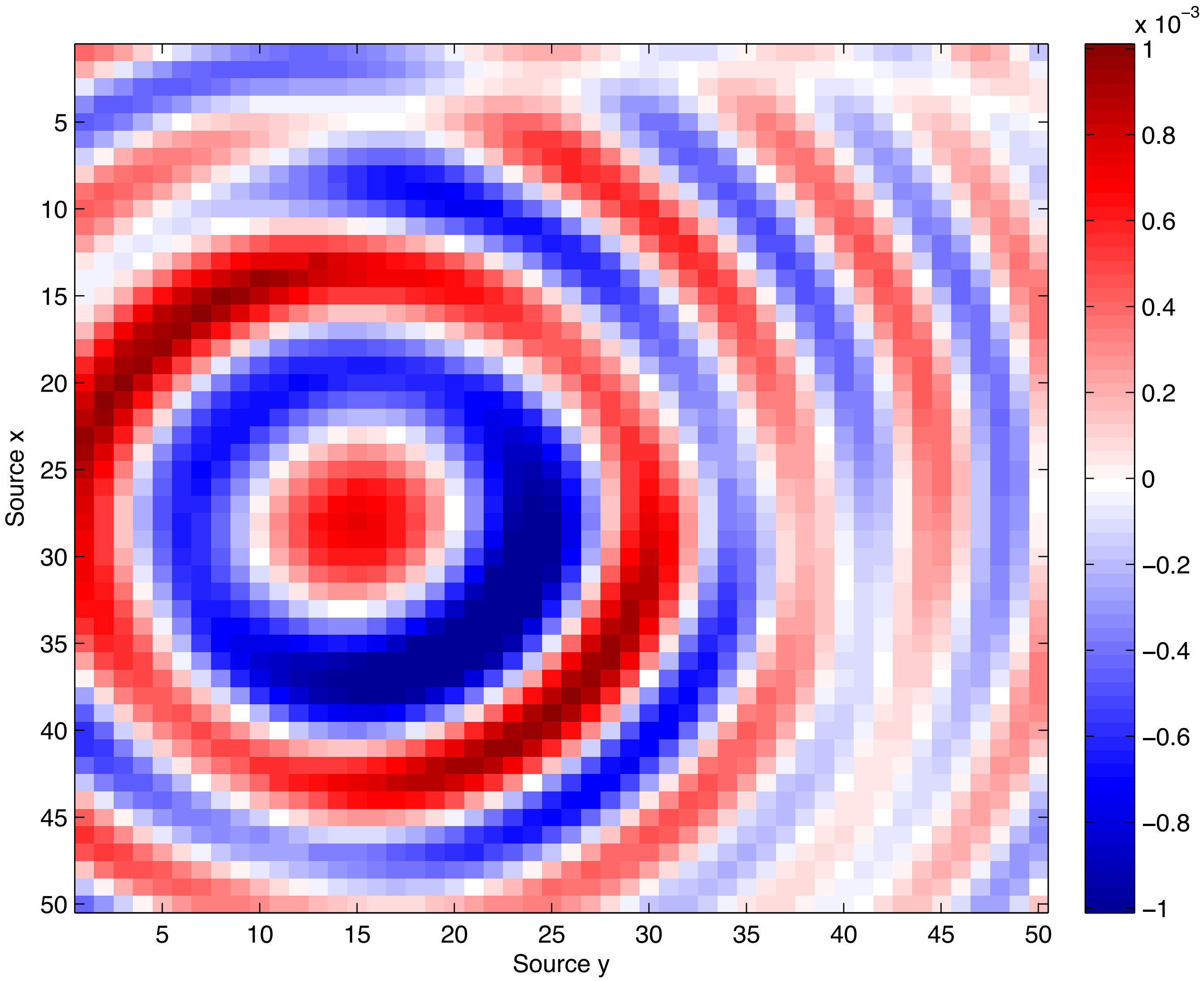


Subsampled Data

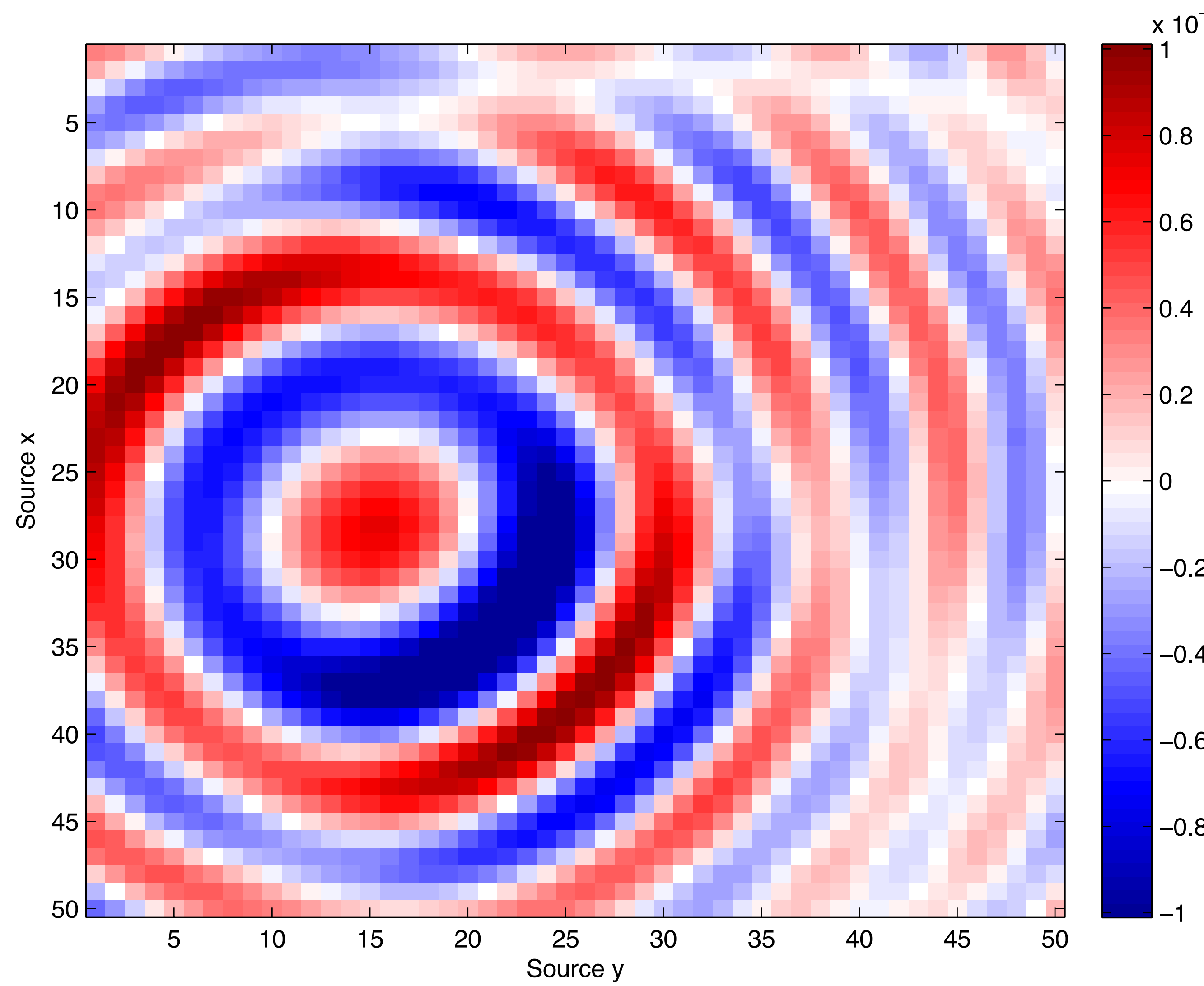
Idealized recovery

75% random entries removed

Common shot gather



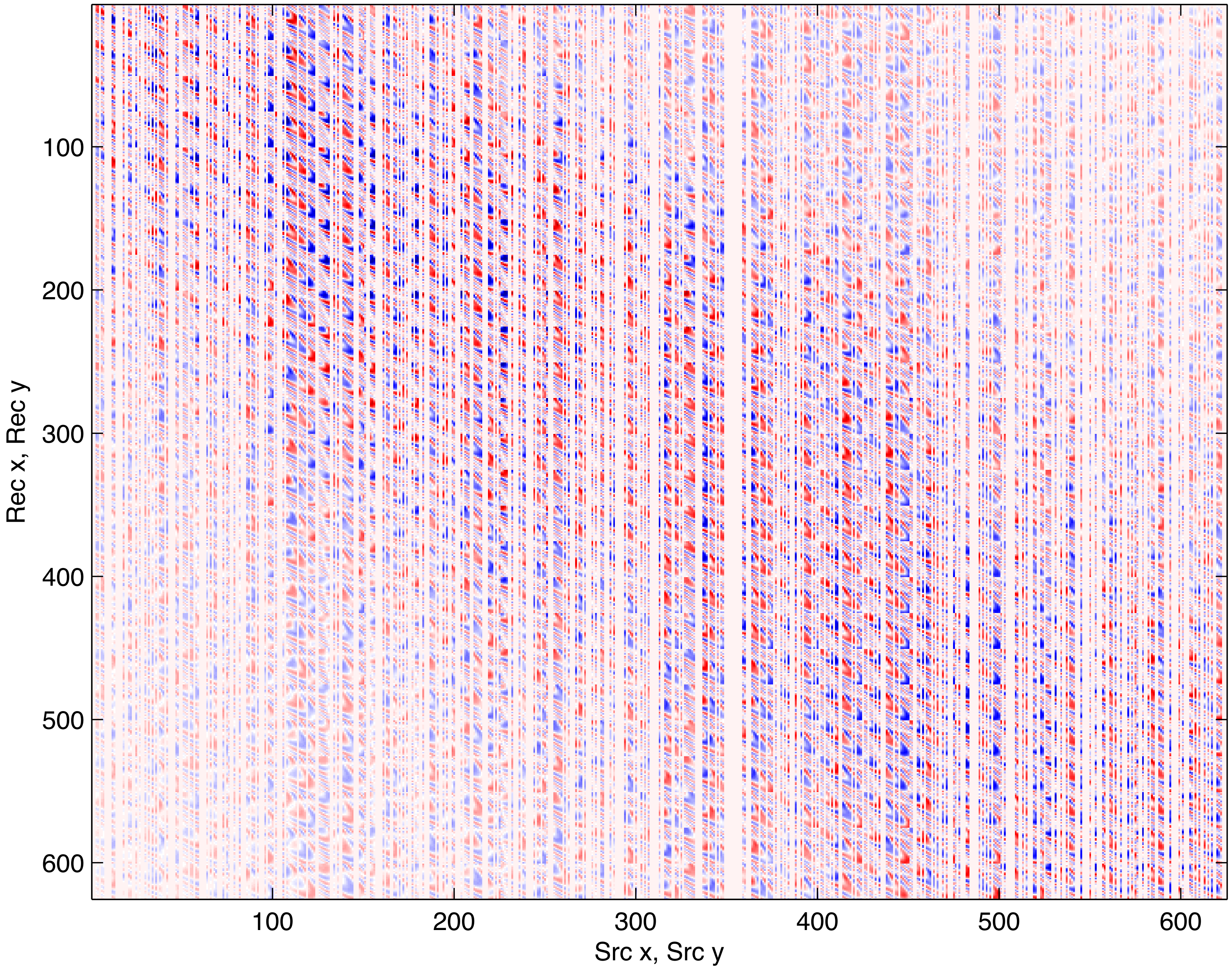
True Data

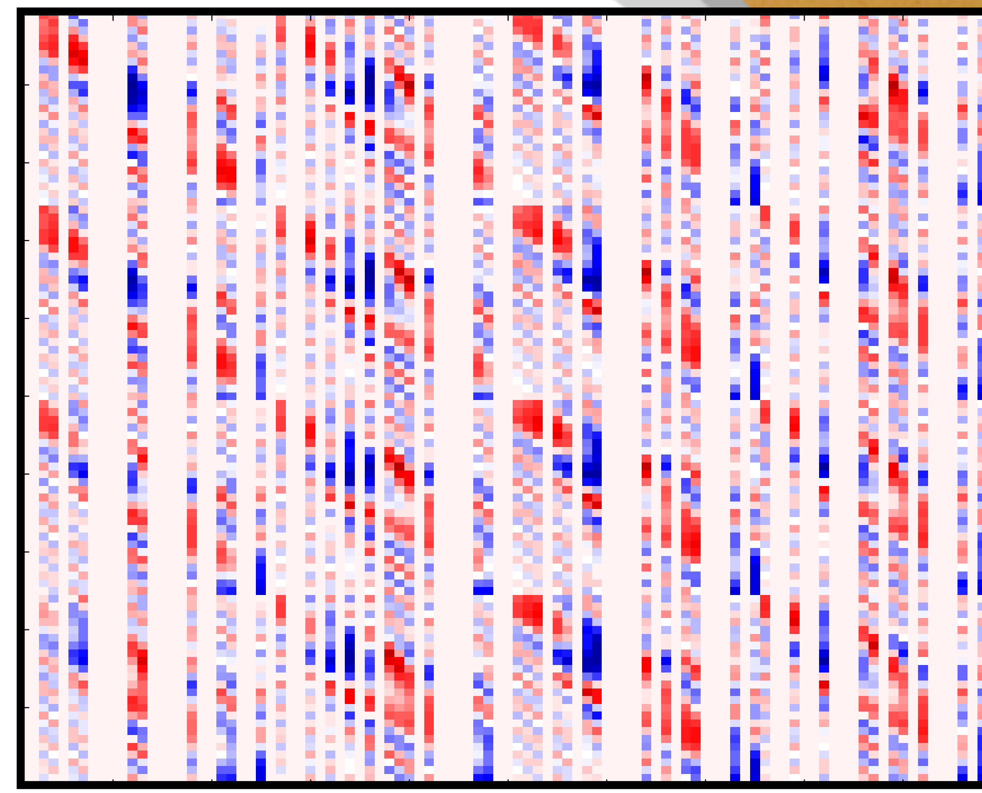


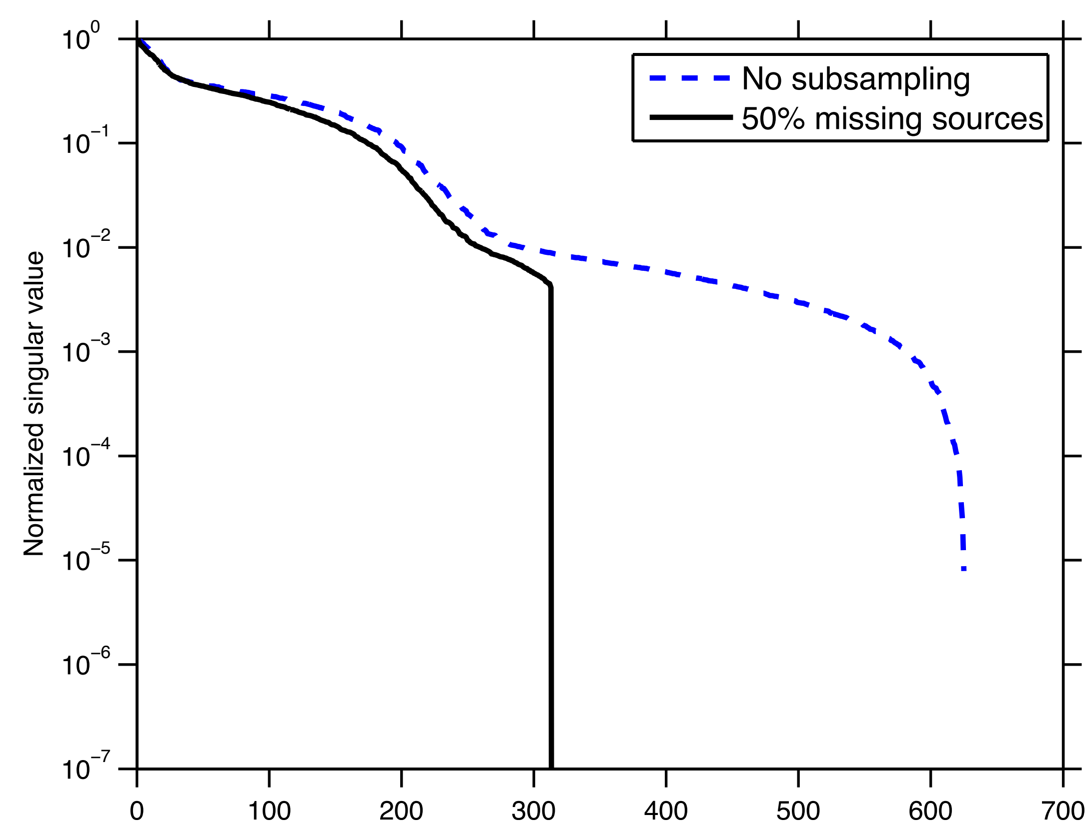
Recovered Shot

SNR 19.3 dB

Sampling



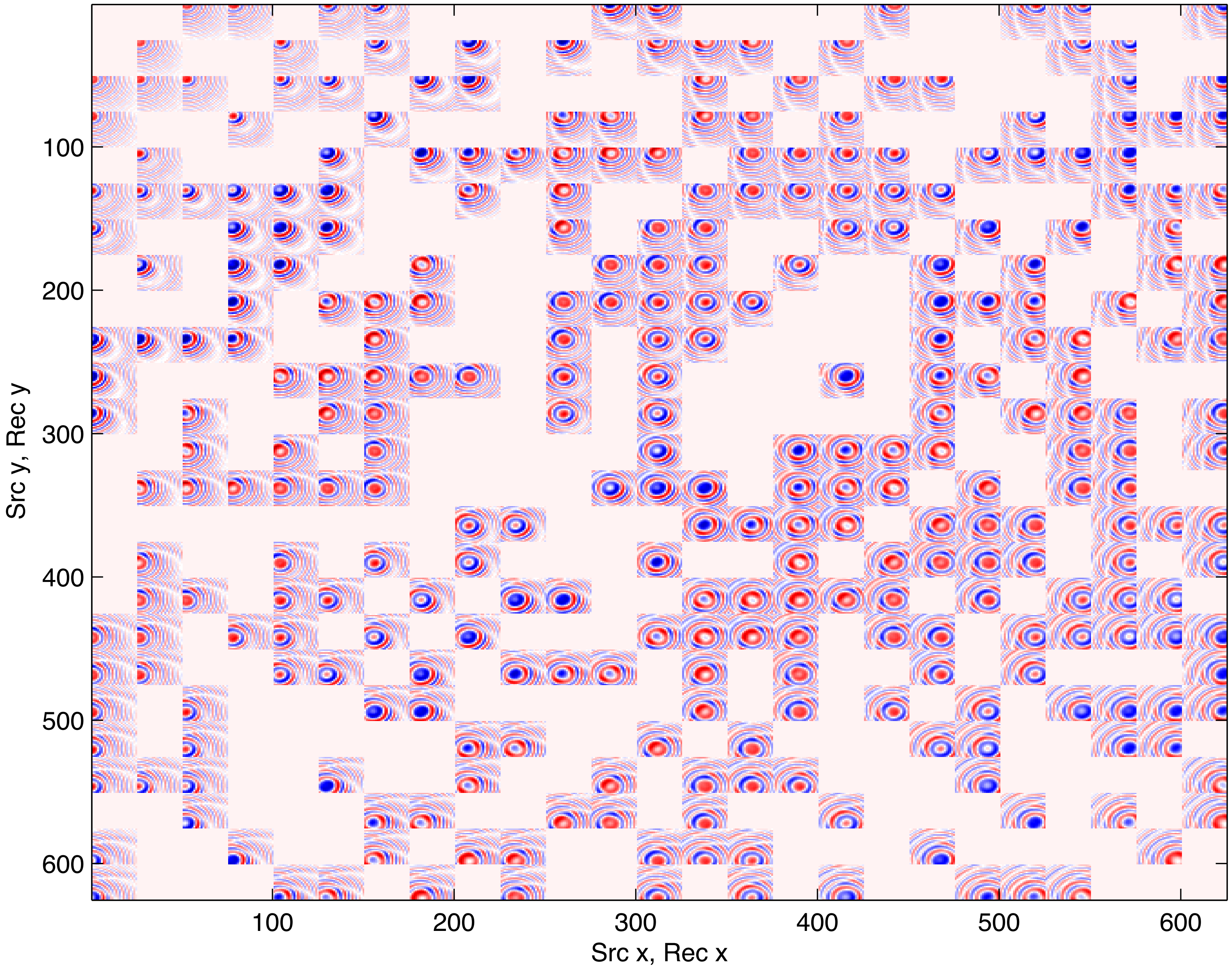


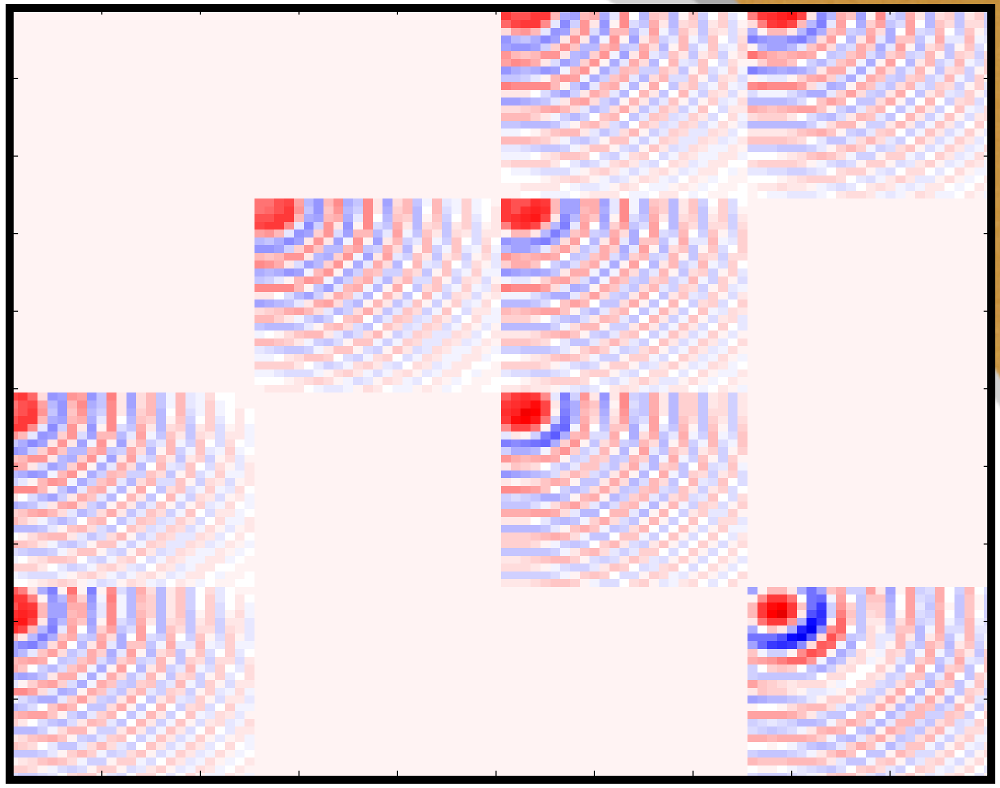


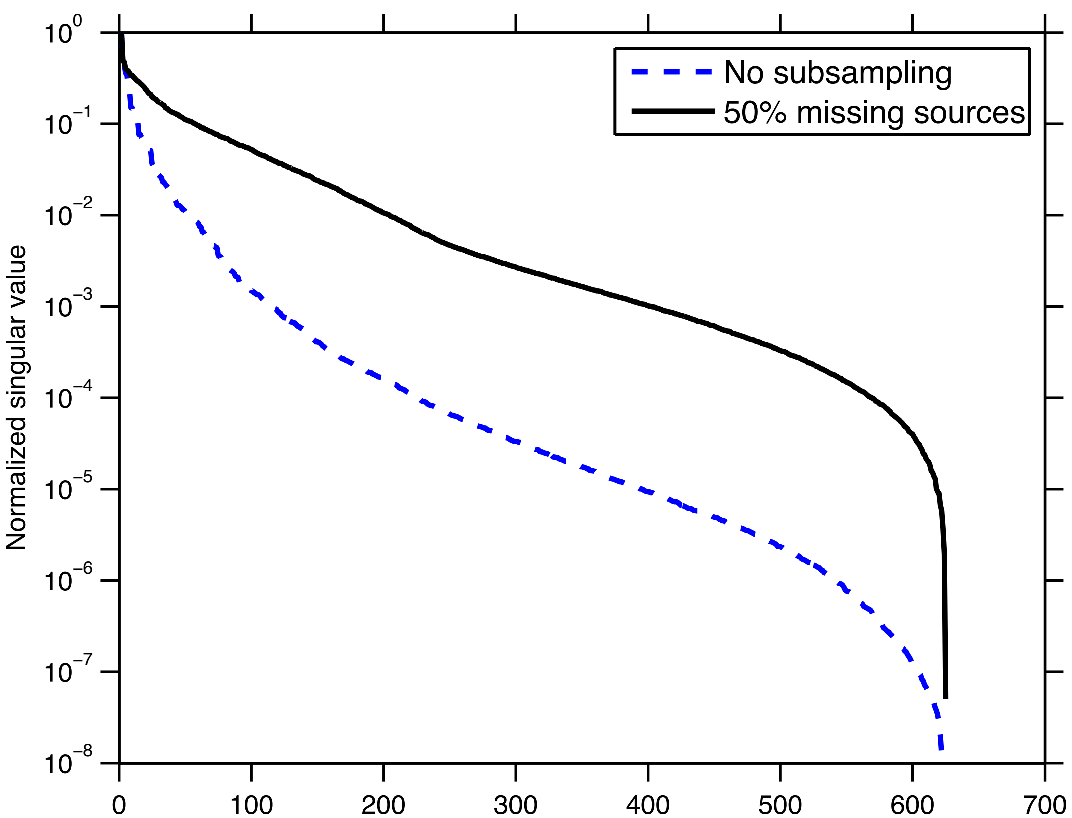
(Rec x, Rec y) matricization

50% randomly missing sources

Sampling







(Src y, Rec y) matricization

50% randomly missing sources

Data organization

- (rec x, rec y) organization
  - High rank
  - Missing sources operator - removes columns
  - Poor recovery scenario

Data organization

- (src y, rec y) decomposition
  - Low rank
  - Missing sources operator - removes blocks
  - Closer to ideal recovery scenario

Multidimensional interpolation

Successful reconstruction scheme

- Signal structure - *Hierarchical Tucker*
- Sampling - *subsampling increases h-rank*
- **Optimization - *fit data in the Hierarchical Tucker format***

Optimization Format

- Given data b with missing sources and/or receivers, subsampling operator A, full tensor expansion operator
$$\phi : (U_t, B_t) \to \mathbb{C}^{n_1 \times \cdots \times n_d}$$
solve
$$\min_{x=(U_t, B_t)} \|A\phi(x) - b\|_2^2$$

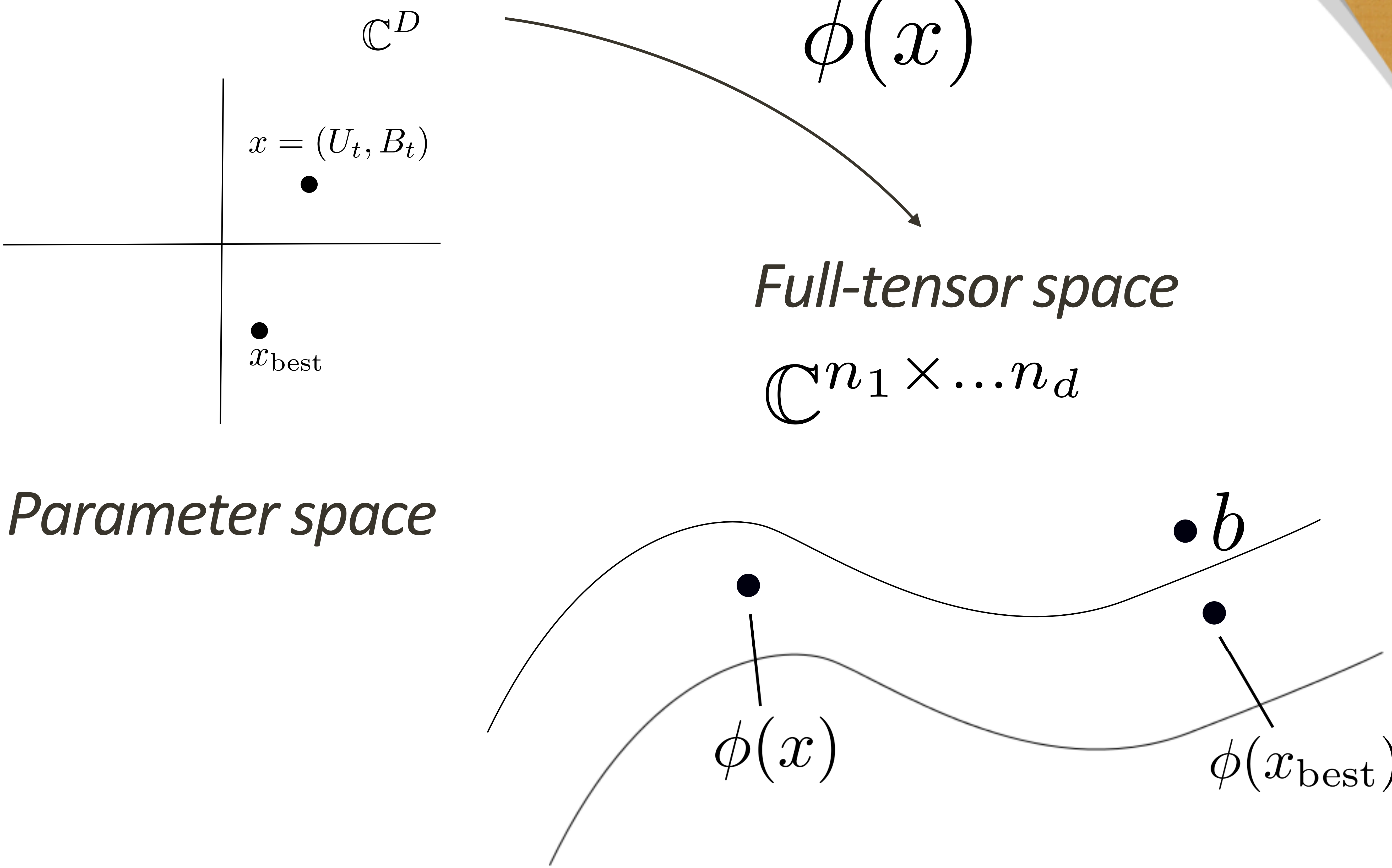
P.A. Absil, R. Mahony, and R. Sepulchre. *Optimization algorithms on matrix manifolds*. Princeton Univ Press, 2008.
A. Uschmajew, B. Vandereycken. *The geometry of algorithms using hierarchical tensors*. 2012

Differential Geometry

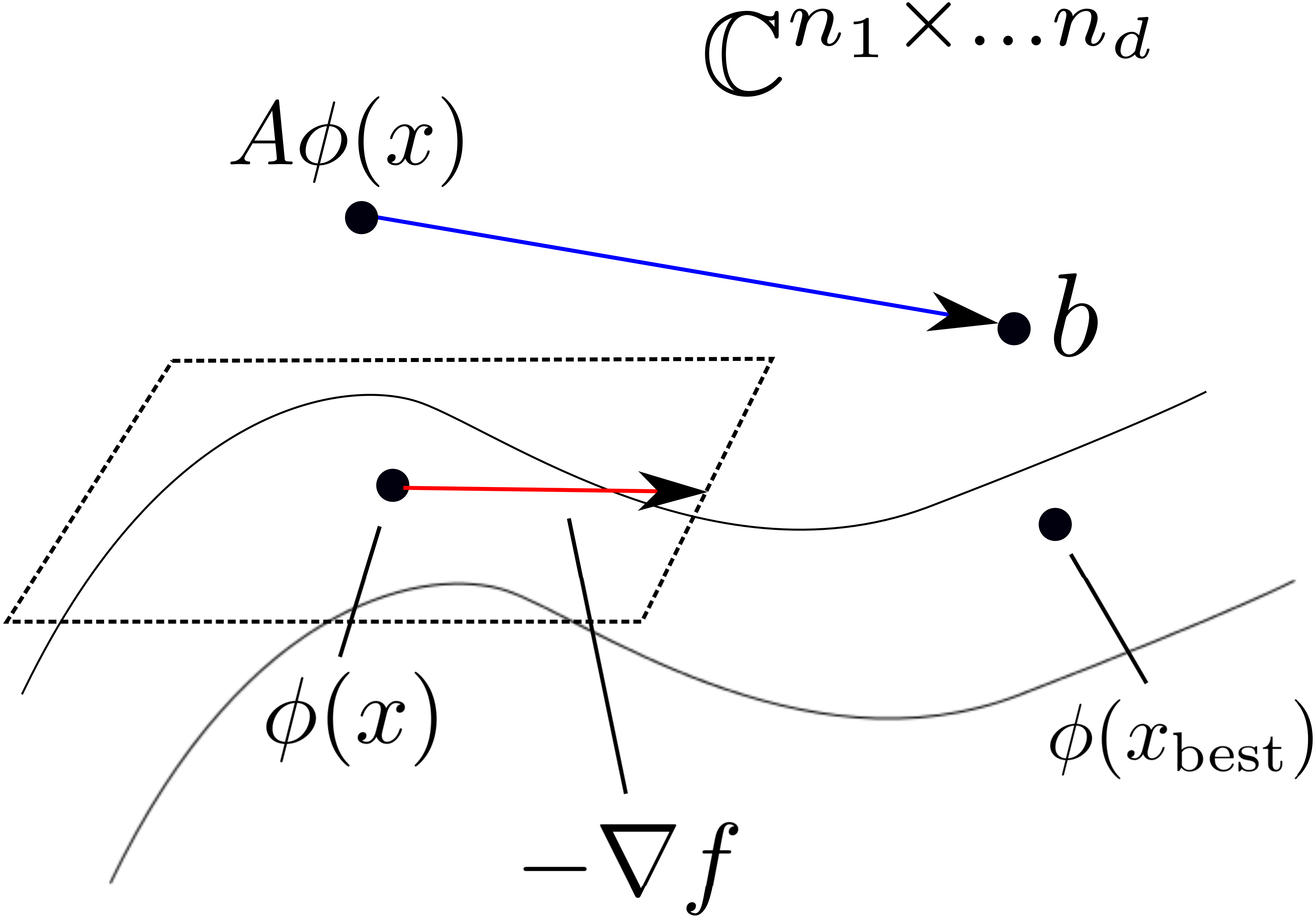
- *The geometry of algorithms using hierarchical tensors* - theoretical analysis of HT tensors
  - Nonlinear, nonconvex space
- Steepest Descent, Conjugate gradient, Gauss-Newton

SLIM

Optimization program



Optimization program



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Derivatives

- Derivatives of a particular node with respect to its children can be computed efficiently
- The chain rule gives the gradient of the function ϕ

Derivative code

```
function [dUdL,dUdR,dUdB] = dHTuck(Uleft, Uright, B)
    Bhat = matricize(B,1,[2 3]);
    Bhat = dematricize(Uright * Bhat,[nr,kl,k],1,[2,3]);
    Bhat = matricize(Bhat,[1 3],2);
    dUdL = opFunction(nl*nr*k, nl*kl, @(x,mode) dUdL_func(x,mode,
Bhat,nr,nl,k,kl));
    BhatR = dematricize( Uleft*matricize(permute(B,[2 1 3]),1, [2 3]),
[nl,kr,k],1,[2 3]);
    dUdR = opKron(matricize(BhatR,[1 3],2),opDirac(nr));
    dUdB = opKron(opDirac(k),Uleft,Uright);
end
function y = dUdL_func(x,mode,Bhat,nr,nl,k,kl)
    if mode == 1
        v = reshape(x,nl,kl);
        q = Bhat * v';
        result = reshape(q,nr,k,nl);
        result = permute(result,[1 3 2]);
        y = vec(result);
    else
        v = reshape(x,nr,nl,k);
        q = Bhat' * matricize(v,[1 3],2);
        q = reshape(q,kl,nl);
        y = vec(q');
    end
end
```

Derivatives

- Only involve matrix-matrix multiplication of small matrices compared to the ambient, full-tensor space
- Immediately parallelizable - Kronecker products done in parallel

Results

Synthetic BG Data

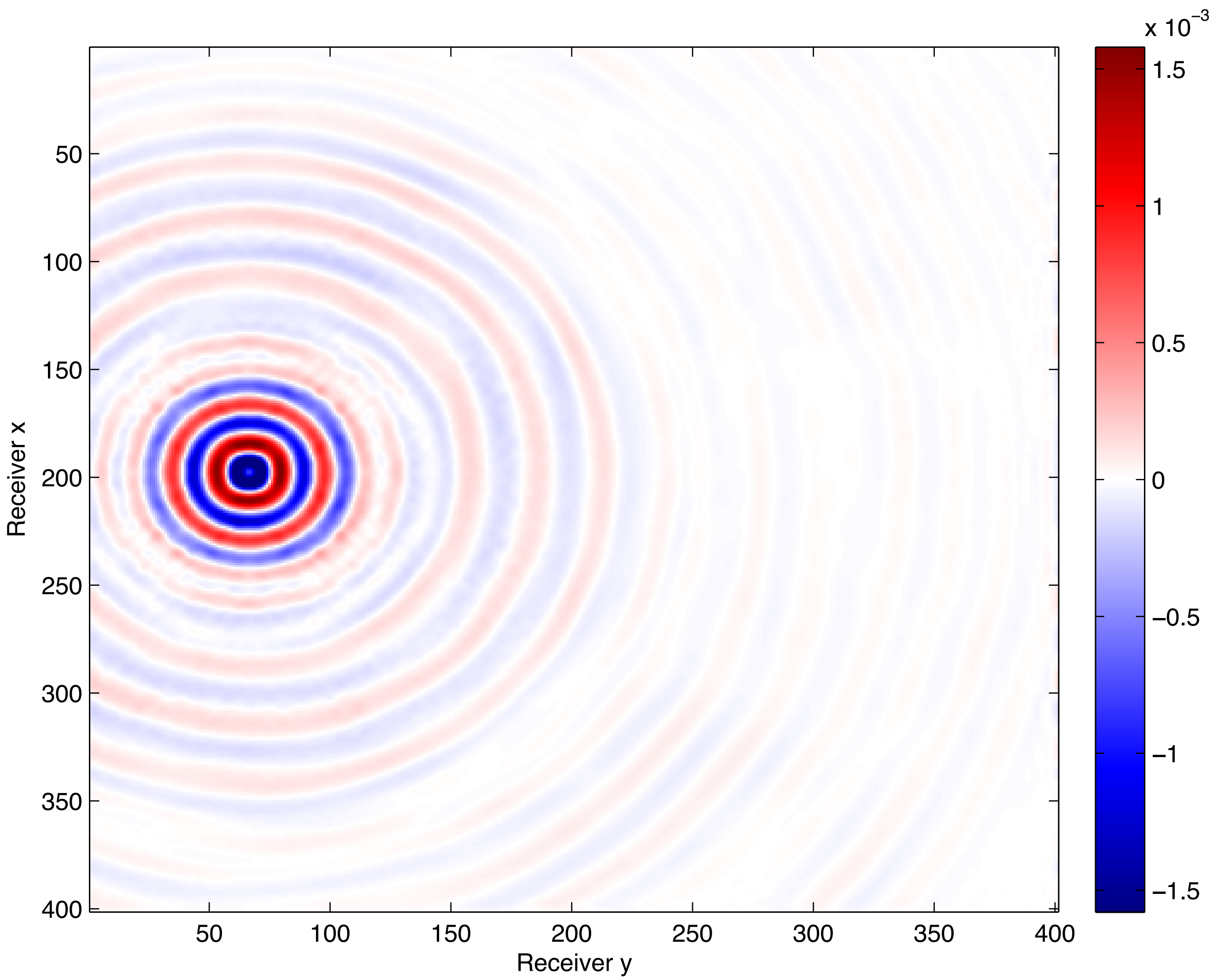
- Unknown model
- 68 x 68 sources with 401 x 401 receivers, data at 4.68Hz, 7.34 Hz
  - Receivers subsampled to 101 x 101, Fourier interpolated back to 401 x 401 to produce figures (due to low frequency content of slice)

Synthetic BG Data

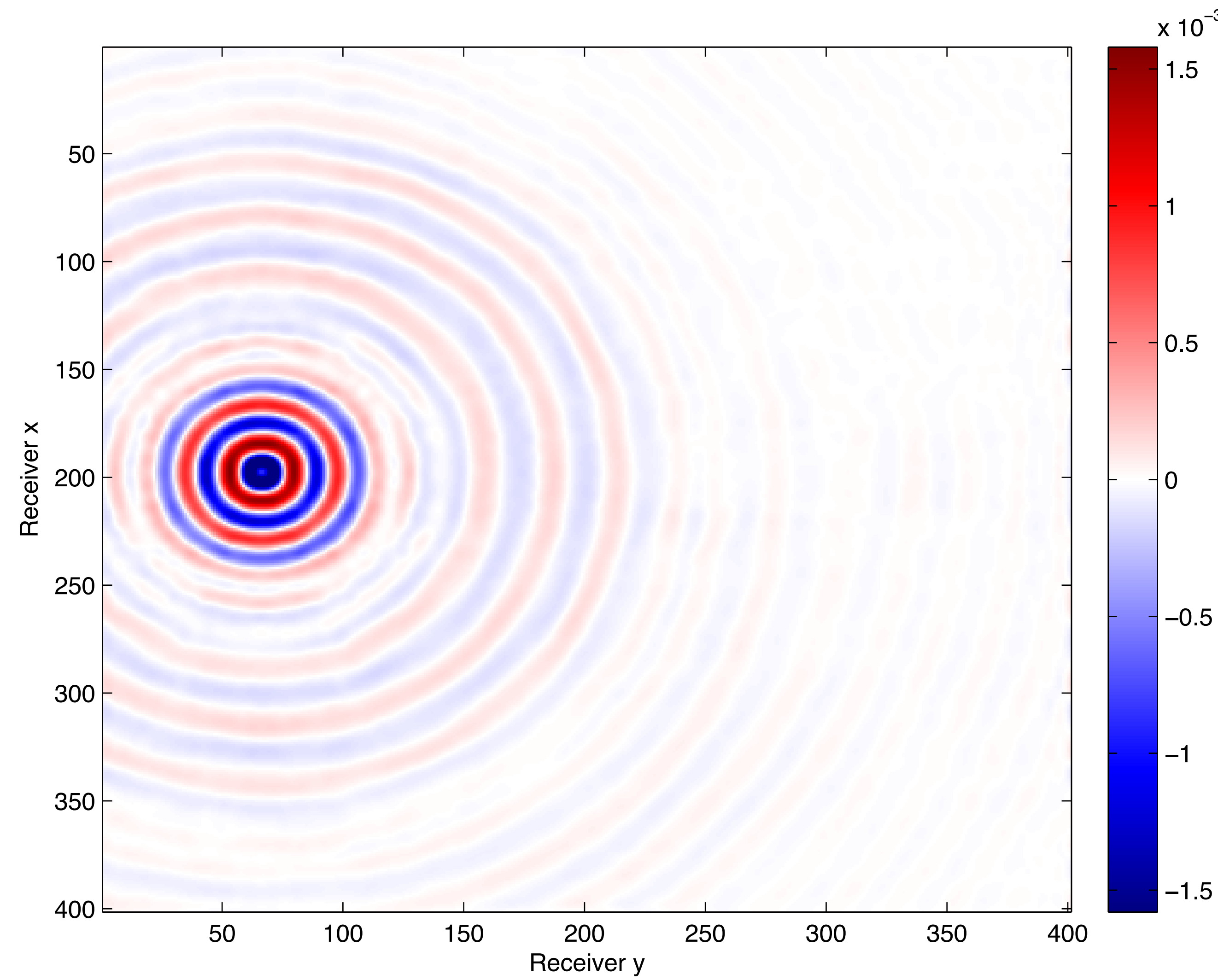
- Single frequency slice (real part) - scaled to unit norm
- Varying percentages of sources have been randomly removed
- Recovered with nonlinear CG

4.86 Hz - 50% missing sources

Common source gather - no data originally



$(x_{src}, y_{src}) = (34,17)$

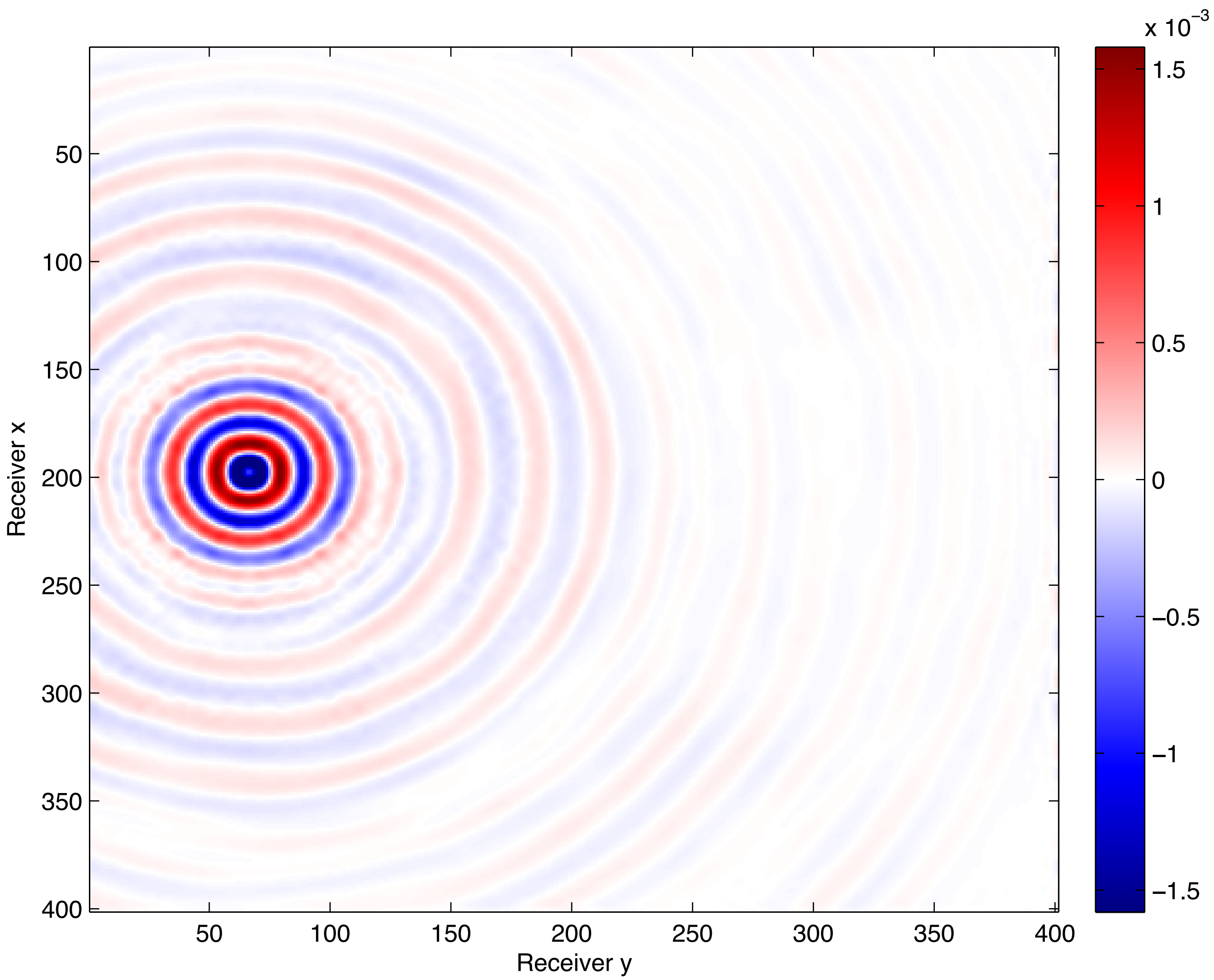


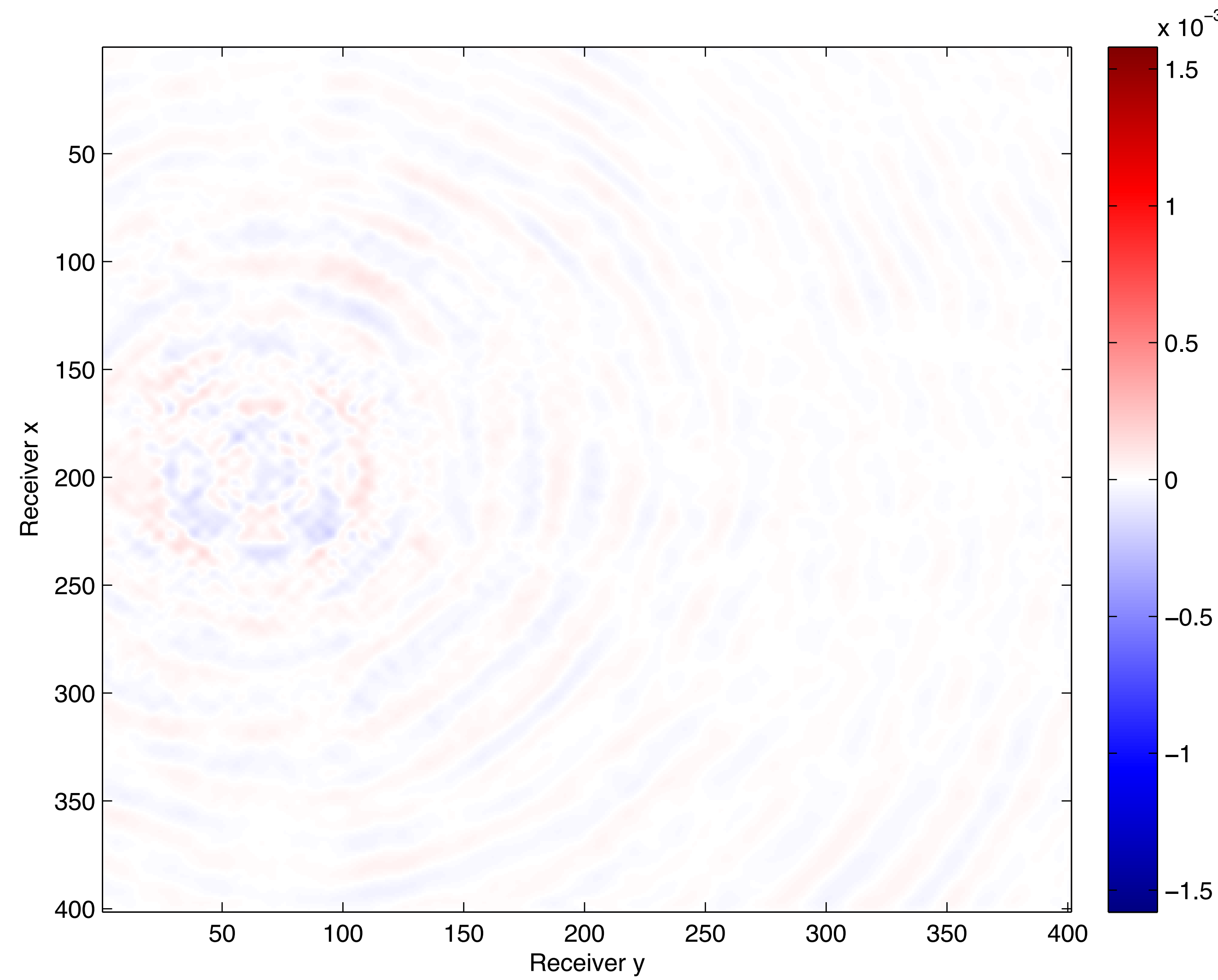
Interpolated Data

SNR 17.5 dB

4.86 Hz - 50% missing sources

Common source gather - no data originally



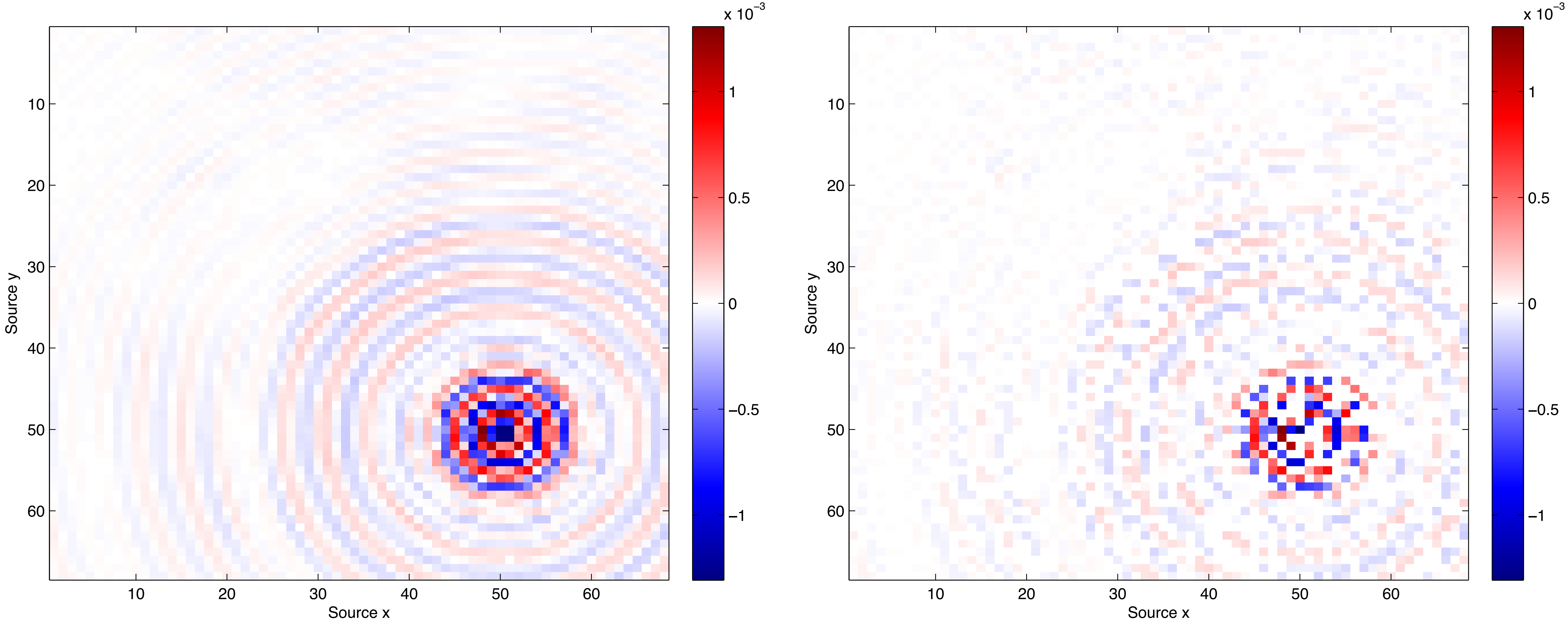


$(x_{src}, y_{src}) = (34,17)$

Difference

4.86 Hz - 50% missing sources

Common receiver gather

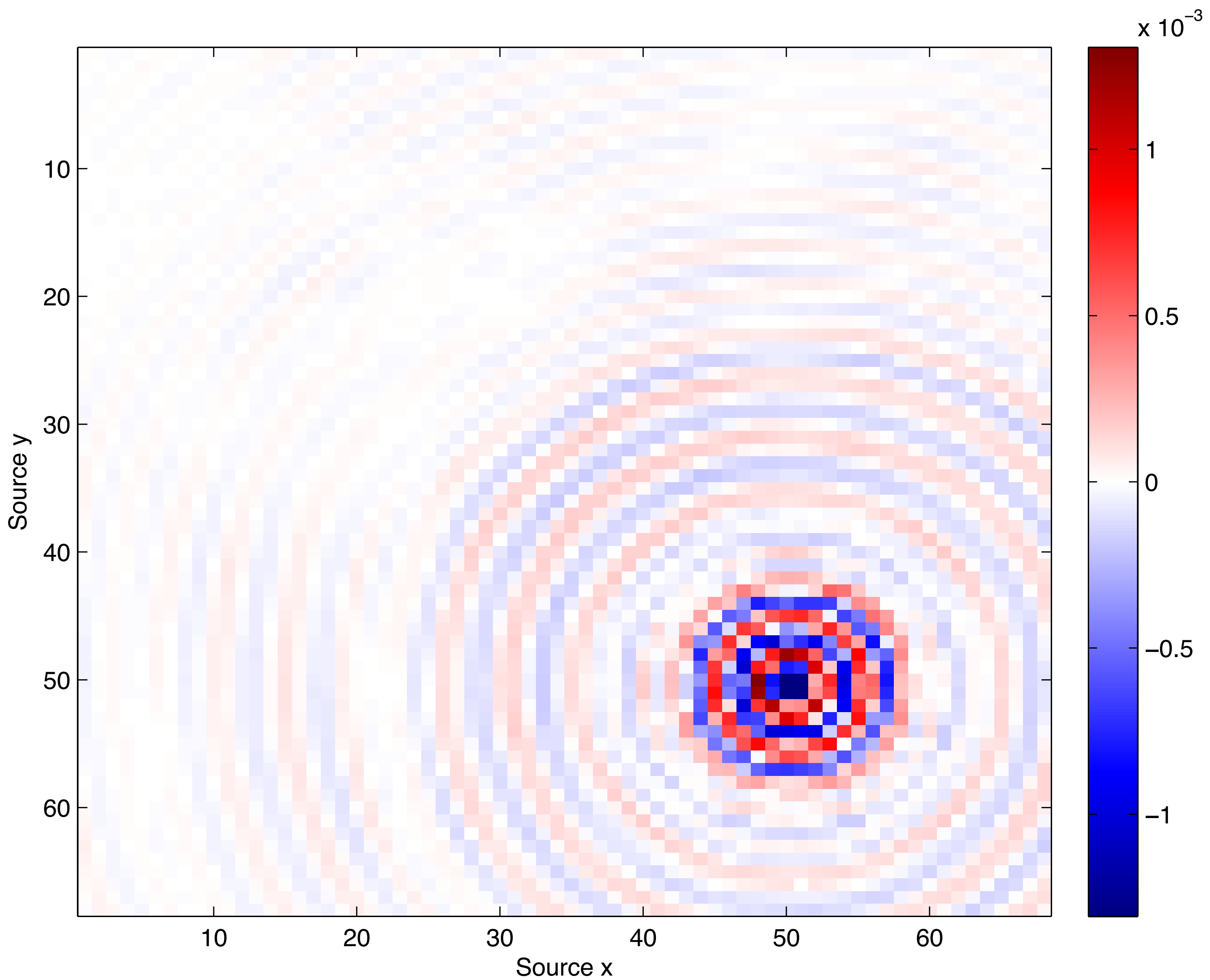


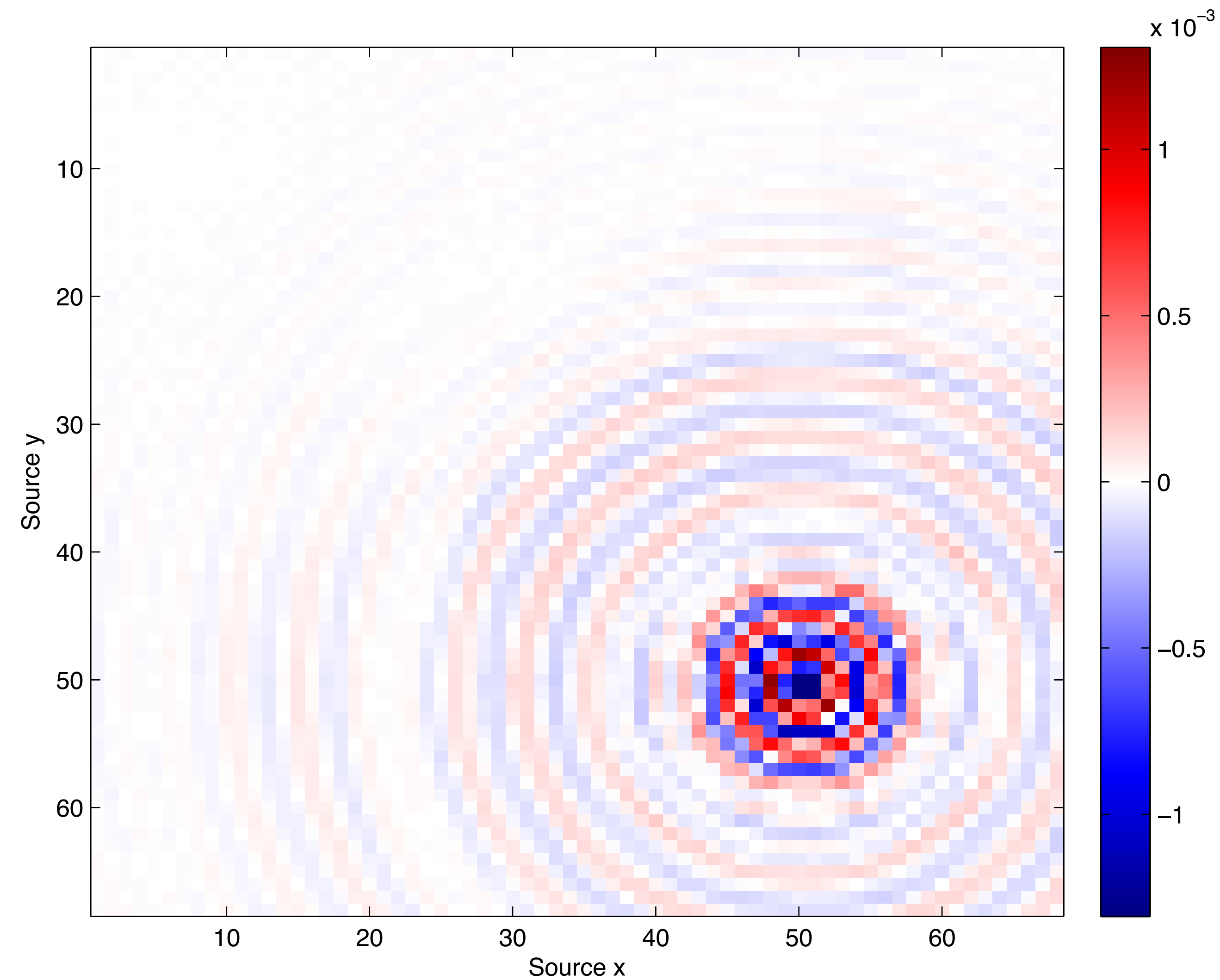
$(x_{rec}, y_{rec}) = (75,75)$

Subsampled Data

4.86 Hz - 50% missing sources

Common receiver gather





$(x_{rec}, y_{rec}) = (75,75)$

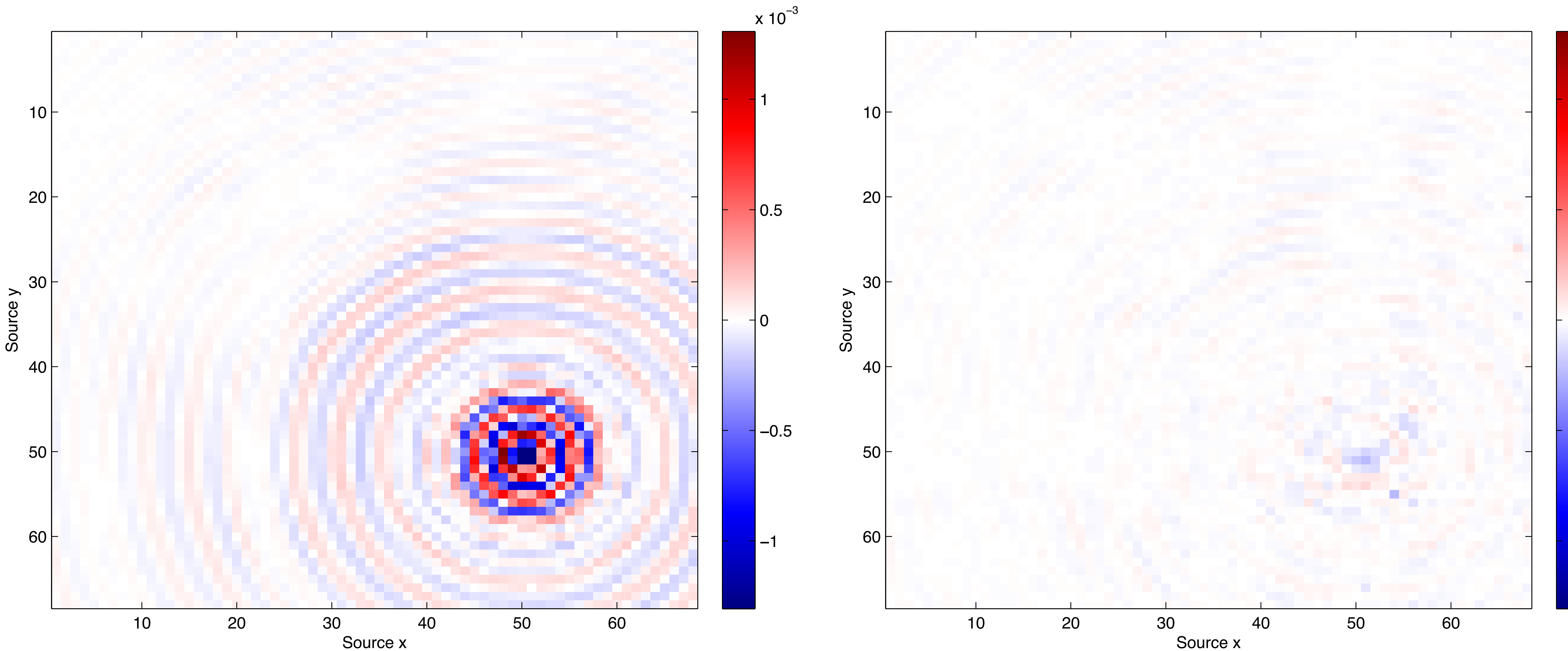
Interpolated Data

SNR 17.3 dB

SLIM

4.86 Hz - 50% missing sources

Common receiver gather



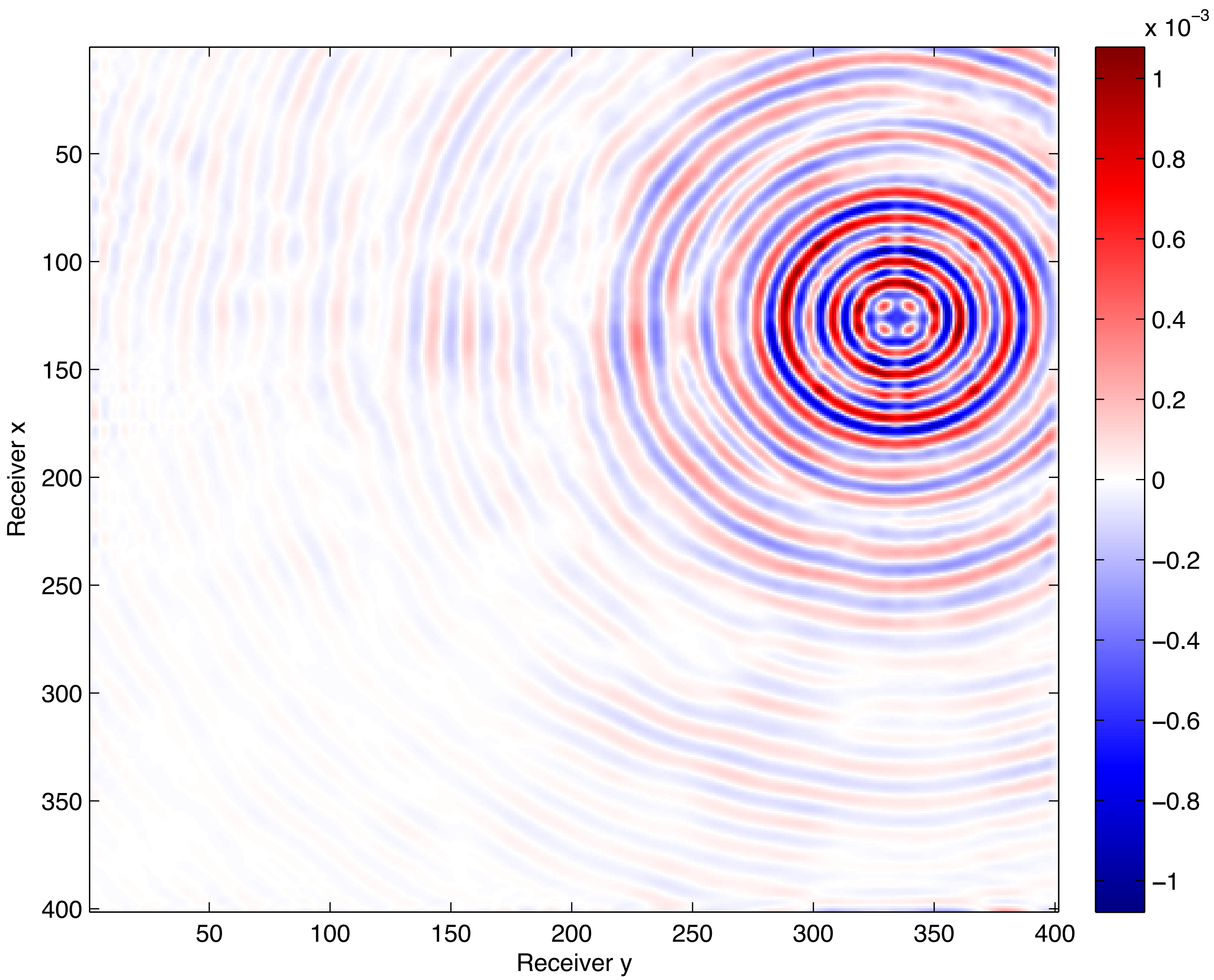
$(x_{rec}, y_{rec}) = (75,75)$

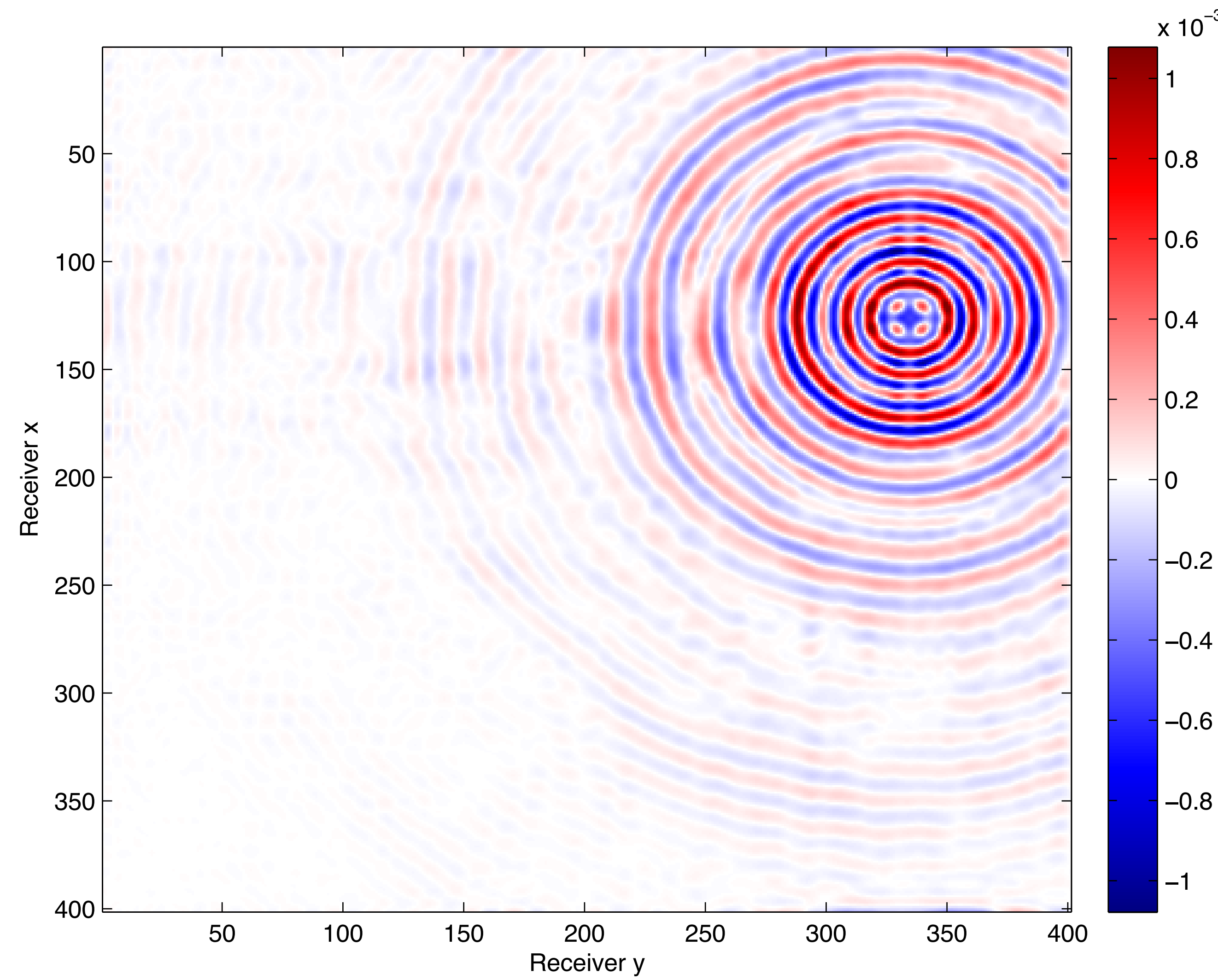
Difference

SLIM

7.34 Hz - 75% missing sources

Common source gather - no data originally





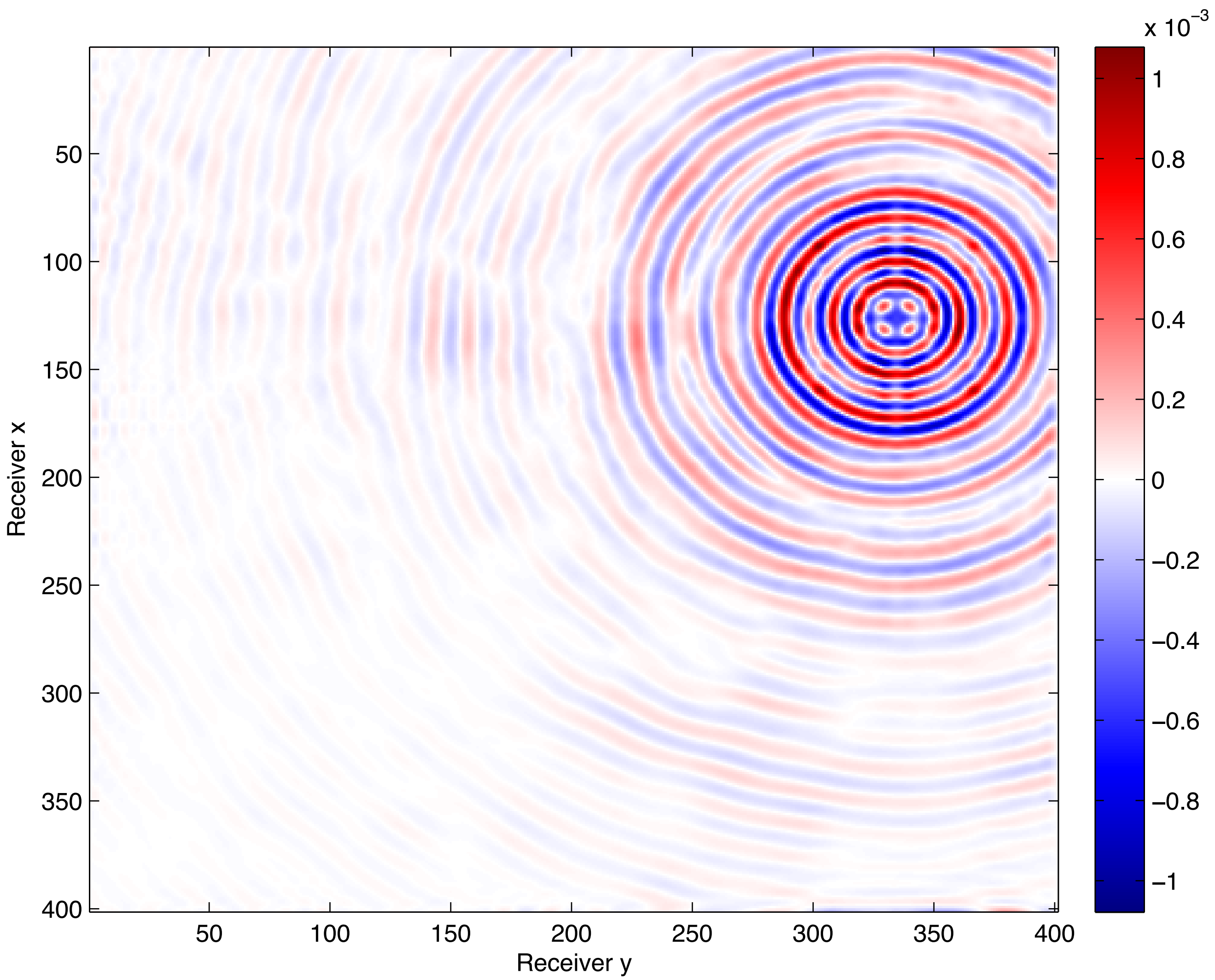
$(x_{src}, y_{src}) = (22,57)$

Interpolated Data

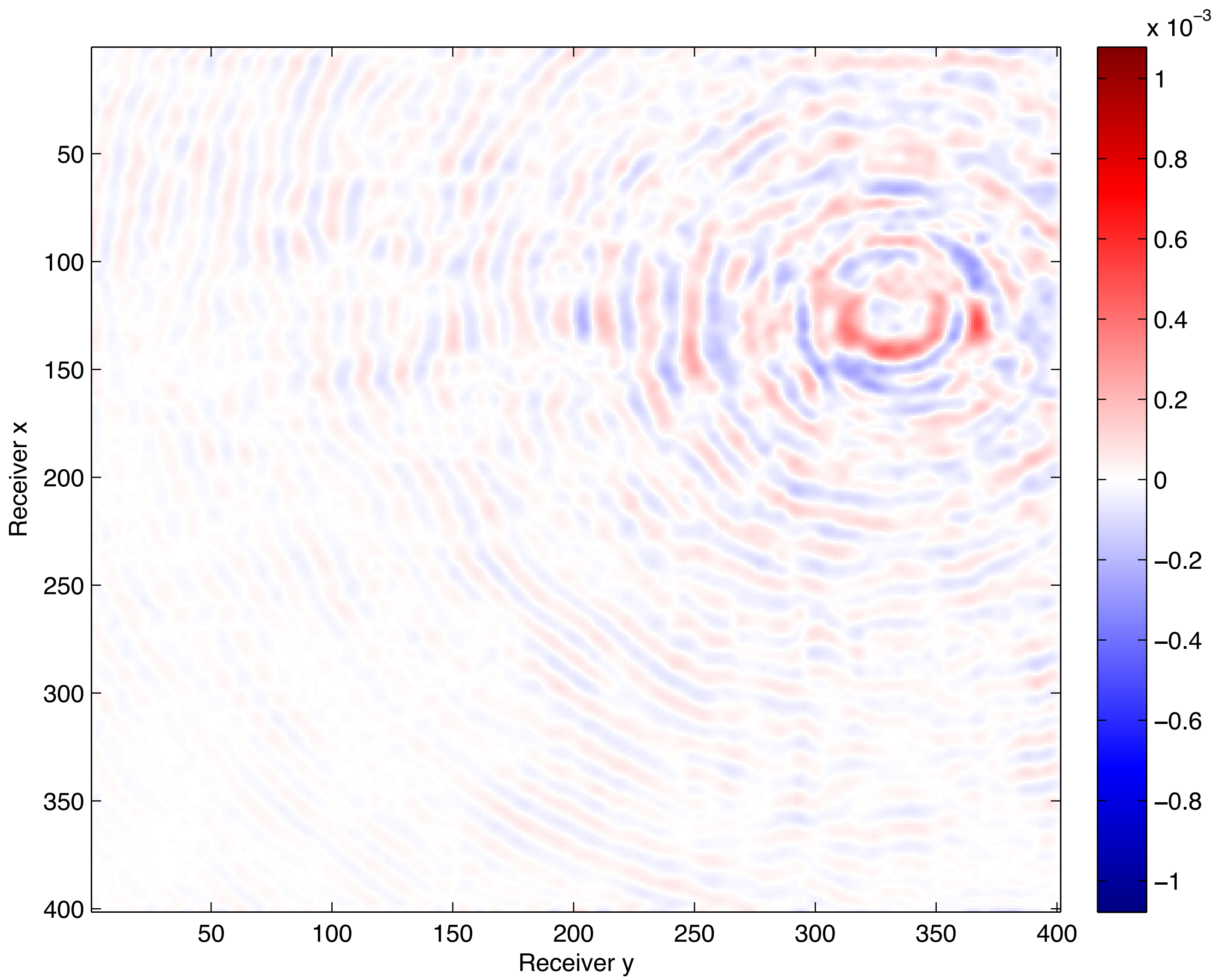
SNR 10.6 dB

7.34 Hz - 75% missing sources

Common source gather - no data originally



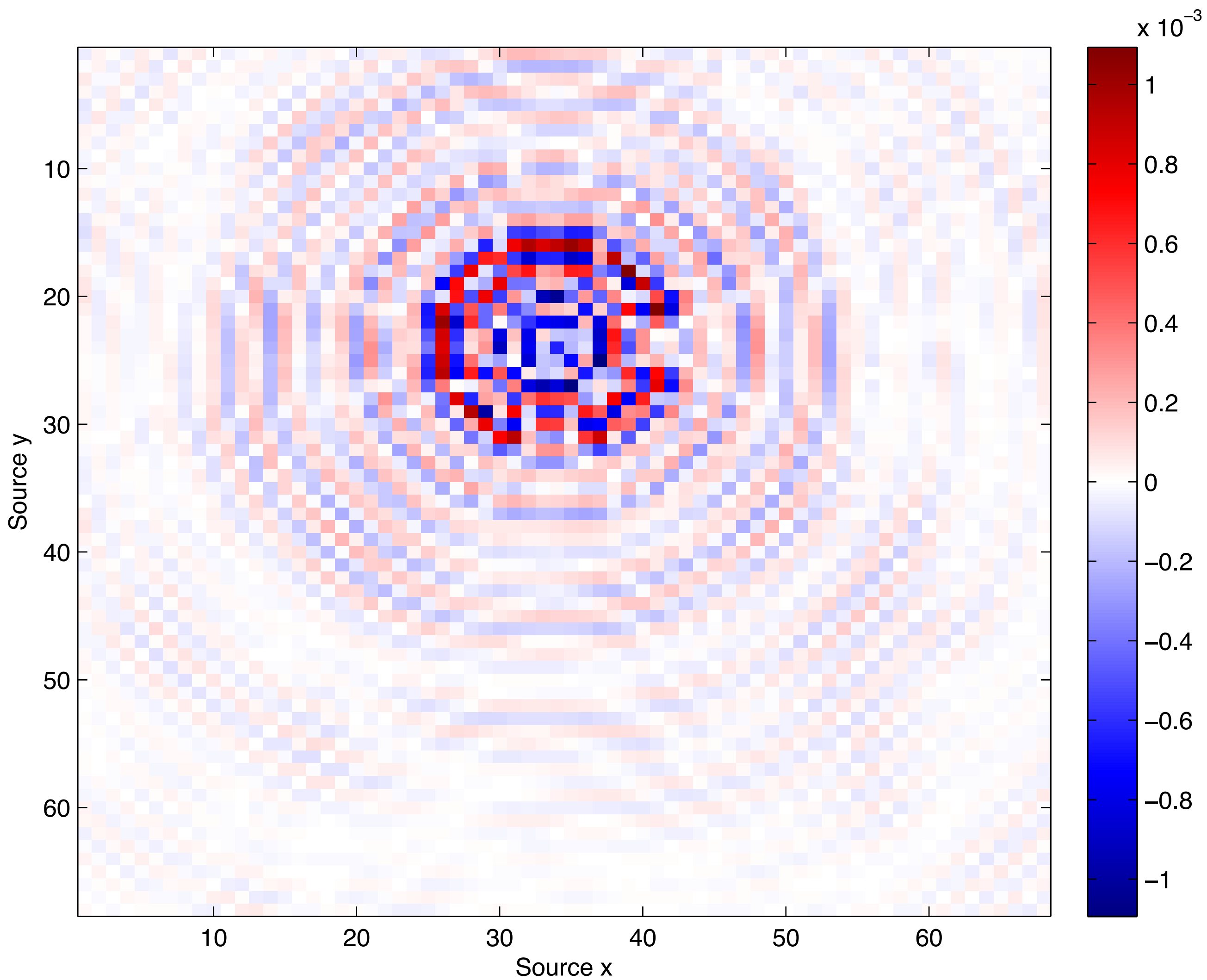
$(x_{src}, y_{src}) = (22,57)$

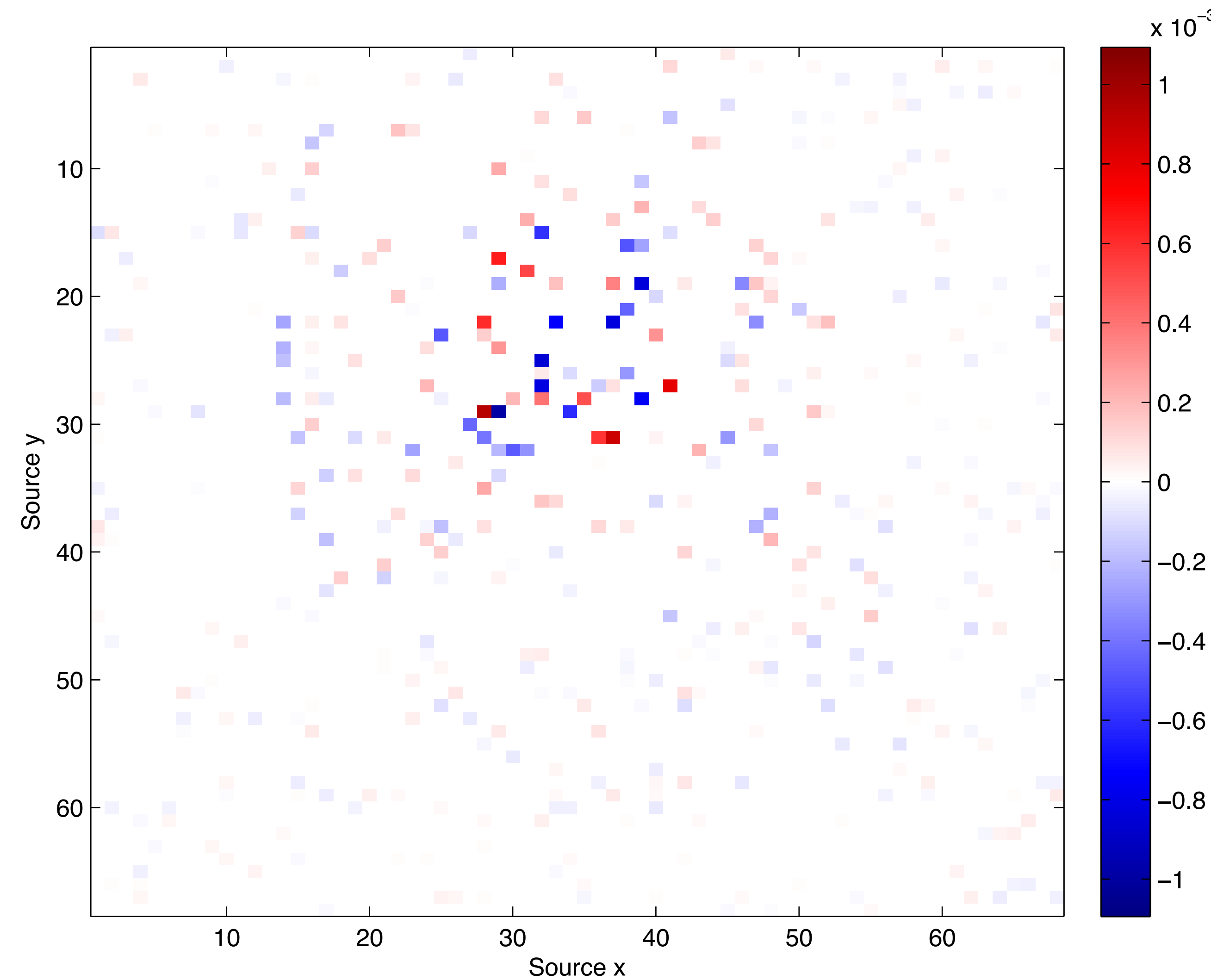


Difference

7.34 Hz - 75% missing sources

Common receiver gather



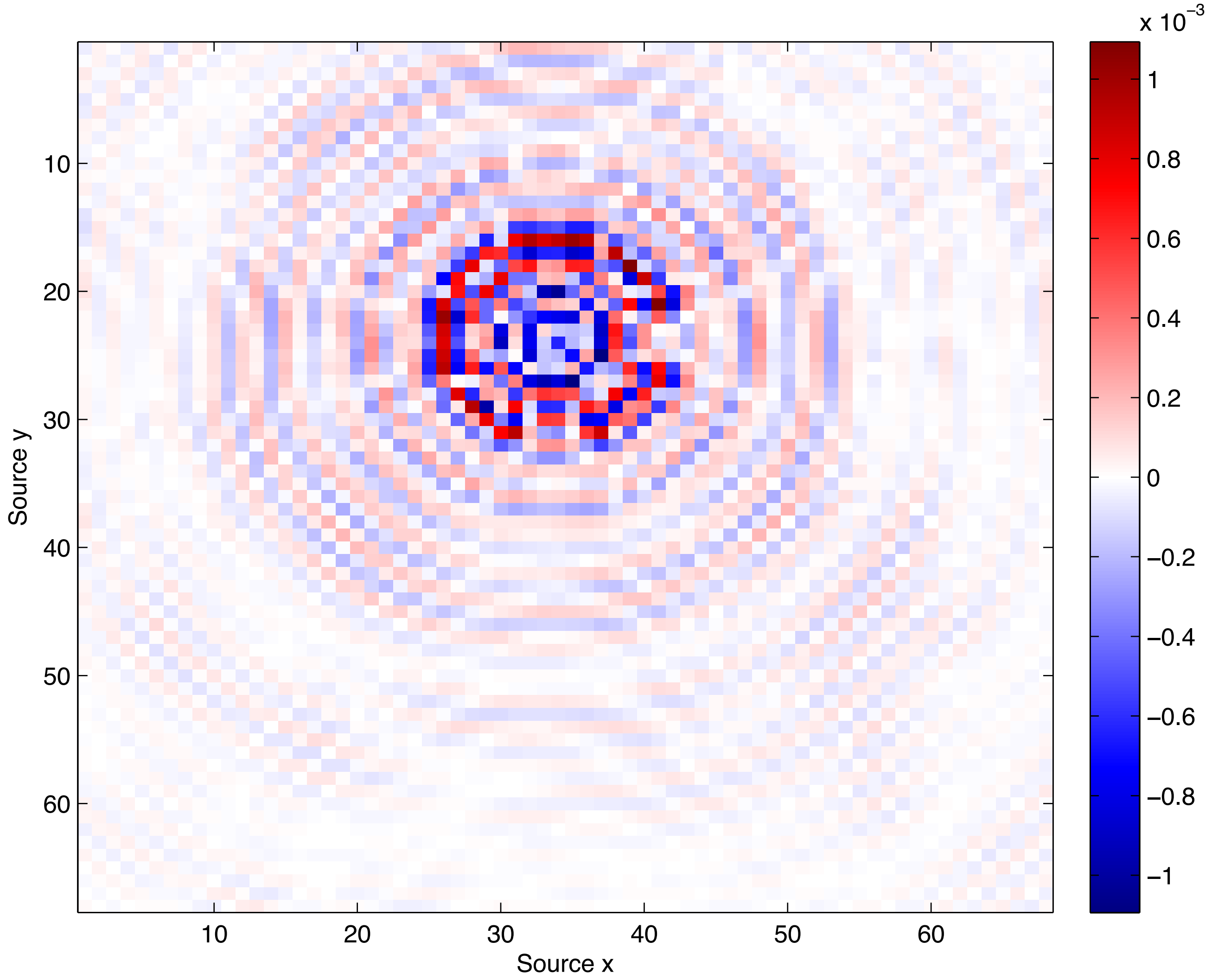


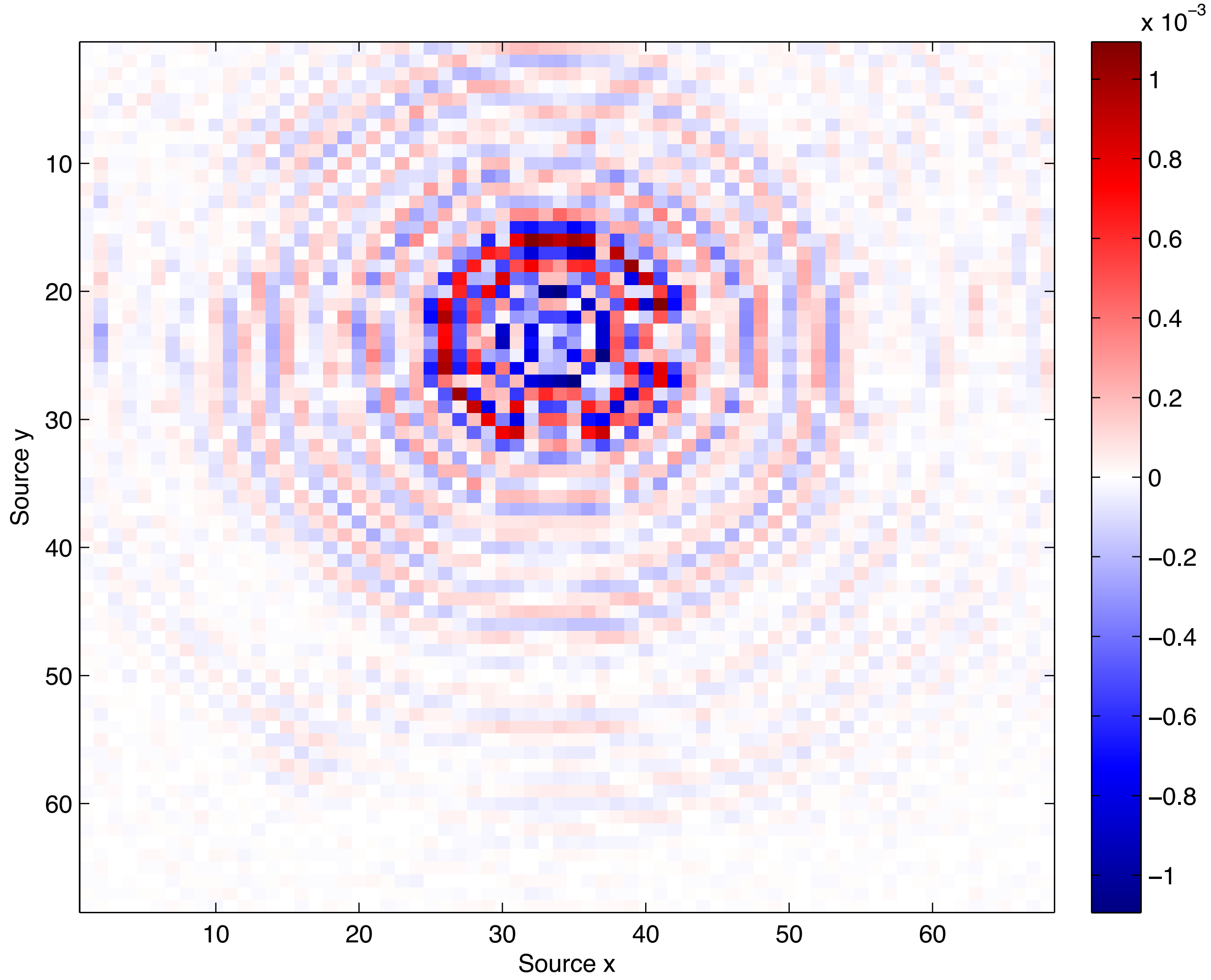
$(x_{rec}, y_{rec}) = (35,50)$

Subsampled Data

7.34 Hz - 75% missing sources

Common receiver gather



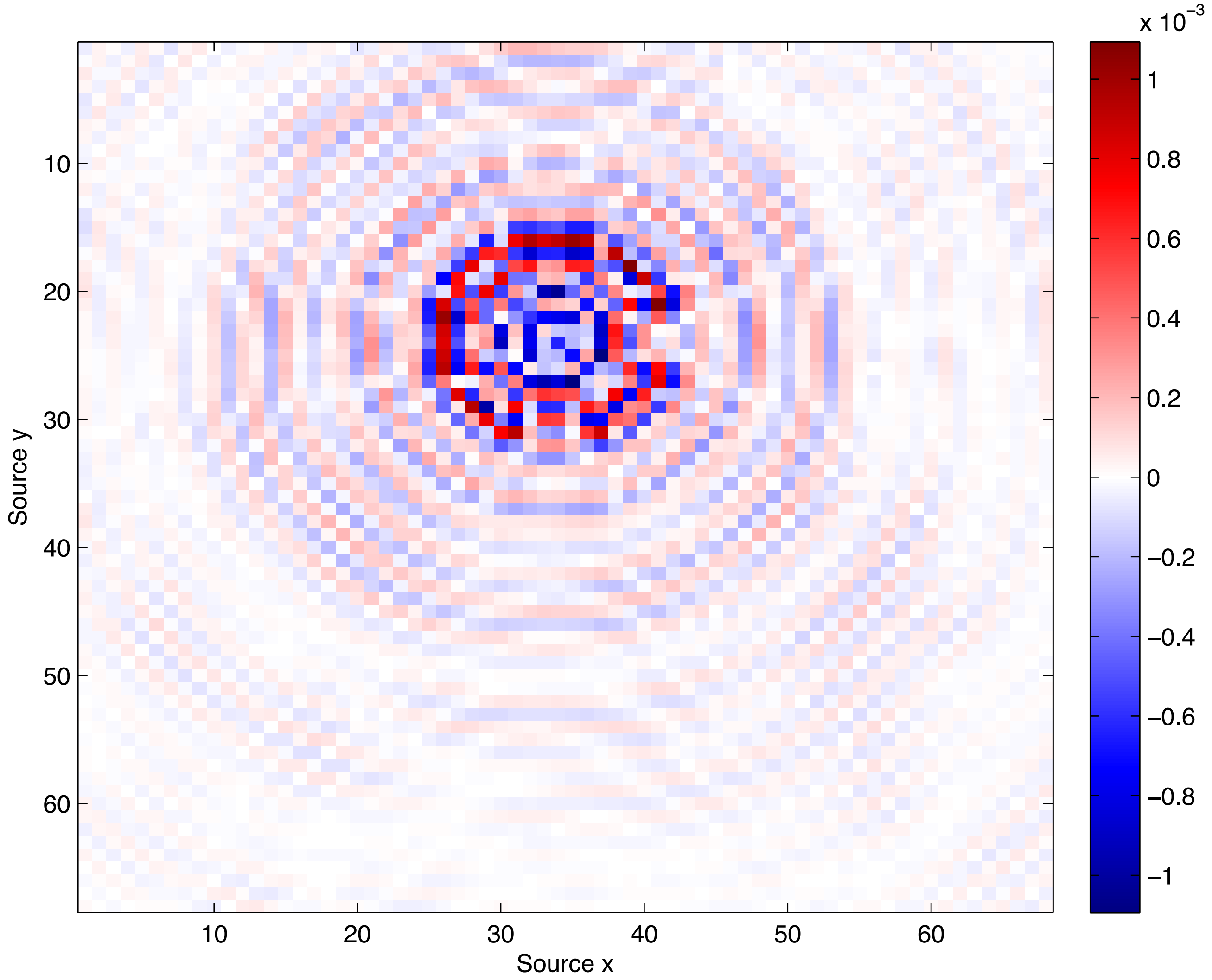


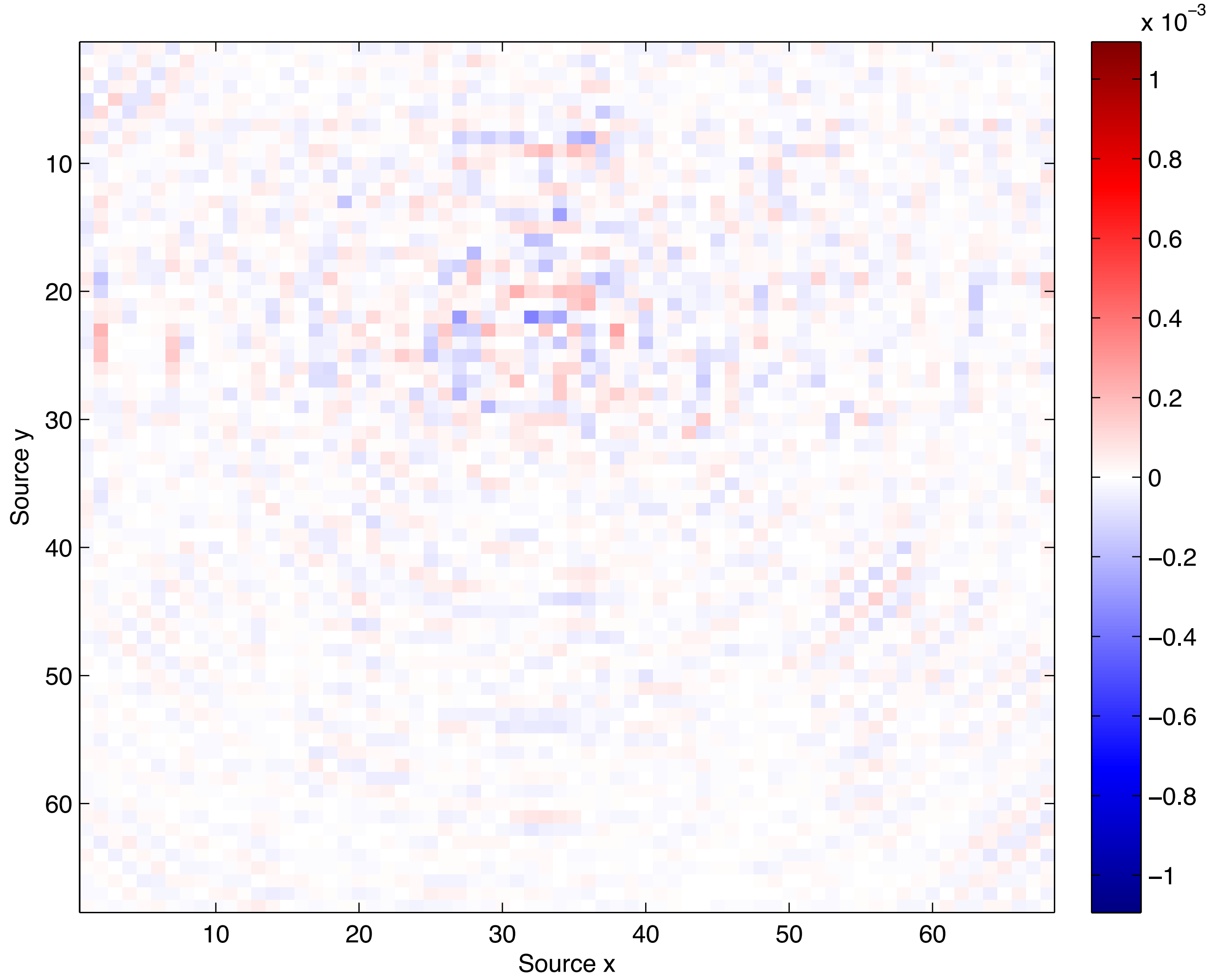
$(x_{rec}, y_{rec}) = (35,50)$

Interpolated Data
SNR 12.1 dB

7.34 Hz - 75% missing sources

Common receiver gather





$(x_{rec}, y_{rec}) = (35,50)$

Difference

SLIM

Summary - SNR

	% Missing srcs	SNR Recovered (dB)
4.86 Hz	25%	18.9
	50%	17.0
	75%	16.2
7.34 Hz	25%	14.4
	50%	14.1
	75%	11.9

Conclusion

- 3D seismic data has an underlying structure that we can exploit for interpolation (Hierarchical Tucker format)
- Different schemes for organizing data - important for recovery

Conclusion

- We can interpolate HT tensors with missing entries using the *Riemannian manifold* structure of the HT format
- Achieve good results from largely subsampled data (75% missing sources)

Acknowledgements

Thank you for your attention

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