

Solving the Helmholtz equation via row-projections

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- Modelling engine for
3D Frequency-domain FWI:
- *work with few sources/
frequencies at each iteration*
 - *flexibility in type of equation*
 - *robust*
 - *parallel*

3D Helmholtz equation:

- *large, sparse, indefinite system*
- *direct factorization not feasible*
- *`standard' preconditioners often fail*
- *successful preconditioners often tailored to specific wave equation*



fast, complicated,..

VS.



simple, robust, ...

Overview

- Kaczmarz preconditioning
- Examples
- Parallelization
- 3D Benchmark
- Inversion
- Conclusions

Kaczmarz

The Kaczmarz method solves a system $A\mathbf{x} = \mathbf{b}$ by successive row projections

$$\mathbf{x} := \mathbf{x} + \frac{\lambda_i}{\|\mathbf{a}_i\|_2^2} (b_i - \mathbf{a}_i^T \mathbf{x}) \mathbf{a}_i,$$

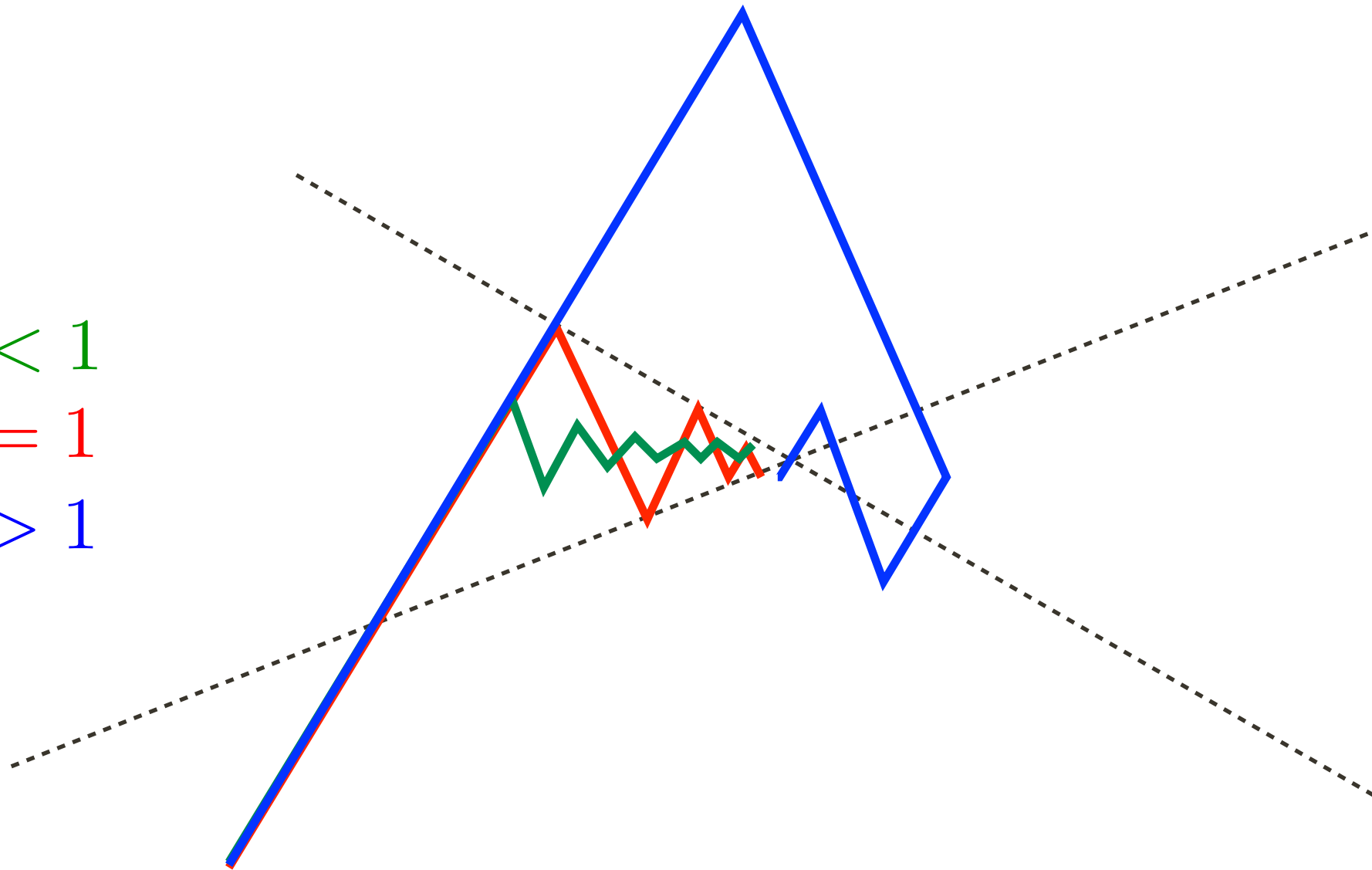
with relaxation parameter $0 < \lambda_i < 2$

Kaczmarz

$$\lambda < 1$$

$$\lambda = 1$$

$$\lambda > 1$$



Kaczmarz

rewrite:

$$\mathbf{x} := \underbrace{\left(I - \frac{\lambda_i}{\|\mathbf{a}_i\|_2^2} \mathbf{a}_i \mathbf{a}_i^T \right)}_{Q_i} \mathbf{x} + \frac{\lambda_i}{\|\mathbf{a}_i\|_2^2} b_i \mathbf{a}_i$$

a double sweep yields

$$\mathbf{x} := \underbrace{(Q_1 Q_2 \dots Q_n Q_n \dots Q_1)}_Q \mathbf{x} + \underbrace{(\dots)}_R \mathbf{b}$$

Kaczmarz

Find a fixed point by solving

$$(I - Q)\mathbf{x} = R\mathbf{b}$$

where $I - Q$ is symmetric and positive semidefinite, so we can use CG (CGMN).

Kaczmarz

We never form the matrix explicitly, but compute its action:

Algorithm 1 DKSWP($A, \mathbf{x}, \mathbf{b}, \lambda$) = $Q\mathbf{x} + R\mathbf{b}$

forward sweep

for $i = 1$ to n **do**

$\mathbf{x} := \mathbf{x} + \lambda(b_i - \mathbf{a}_i^T \mathbf{x})\mathbf{a}_i / \|\mathbf{a}_i\|_2^2$

end for

backward sweep

for $i = n$ to 1 **do**

$\mathbf{x} := \mathbf{x} + \lambda(b_i - \mathbf{a}_i^T \mathbf{x})\mathbf{a}_i / \|\mathbf{a}_i\|_2^2$

end for

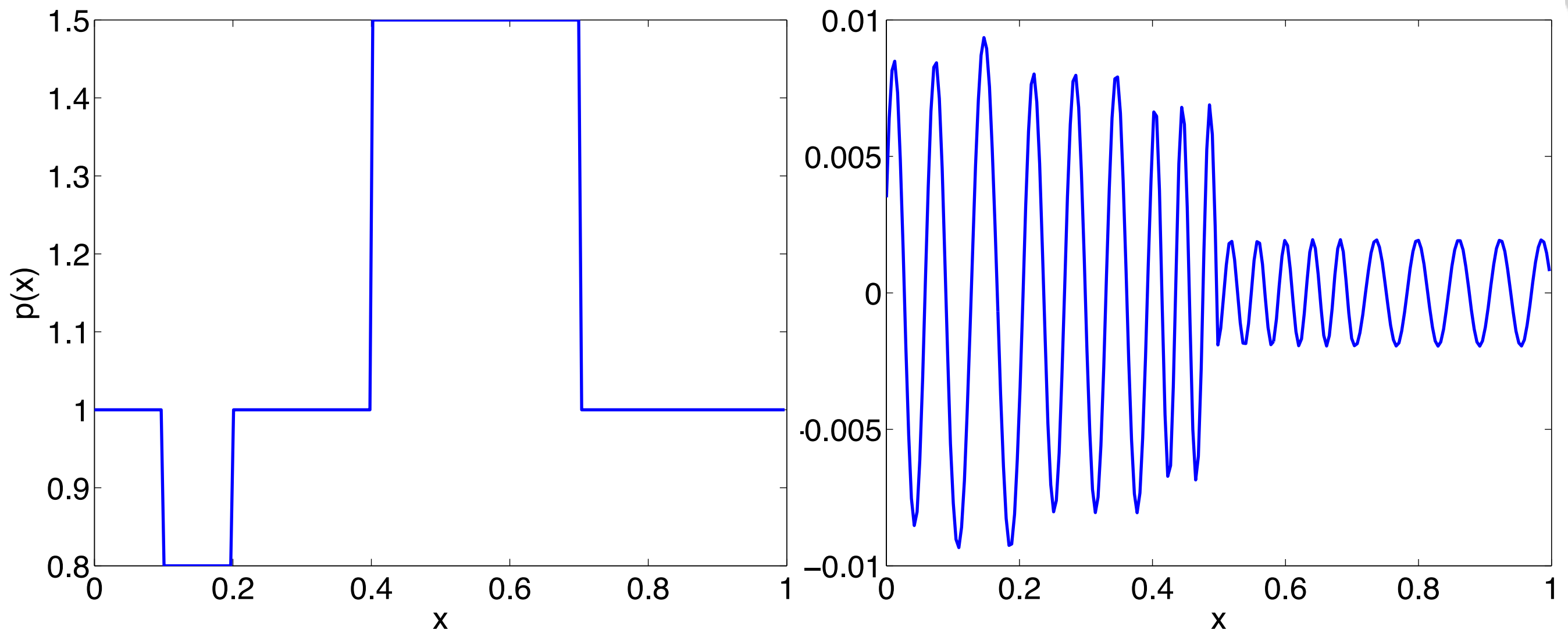
return $\mathbf{x}$

Kaczmarz

- low complexity
- low memory (same as original matrix)
- no setup time

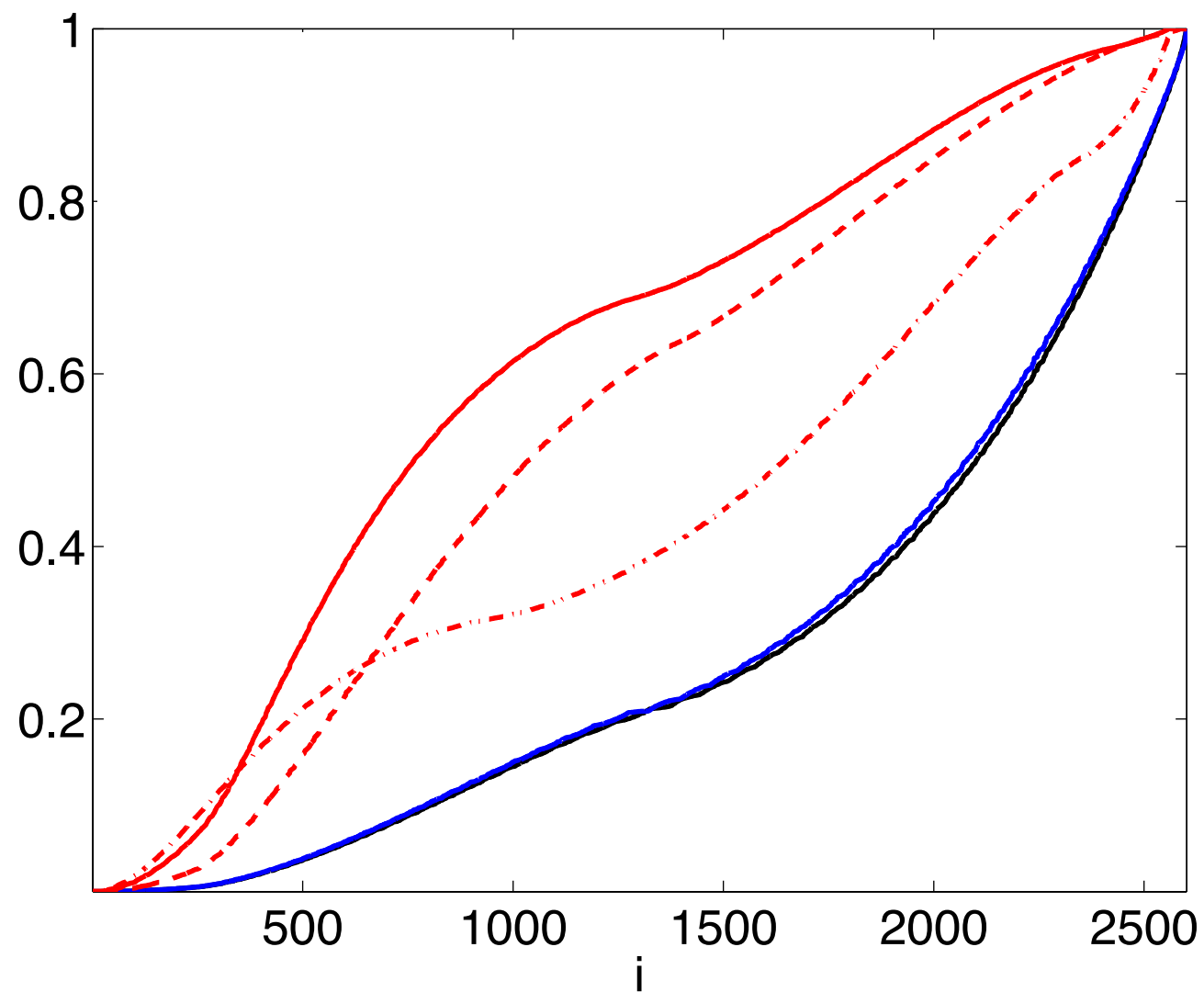
1D results

1D profile, varying k , 10 p/wavelength

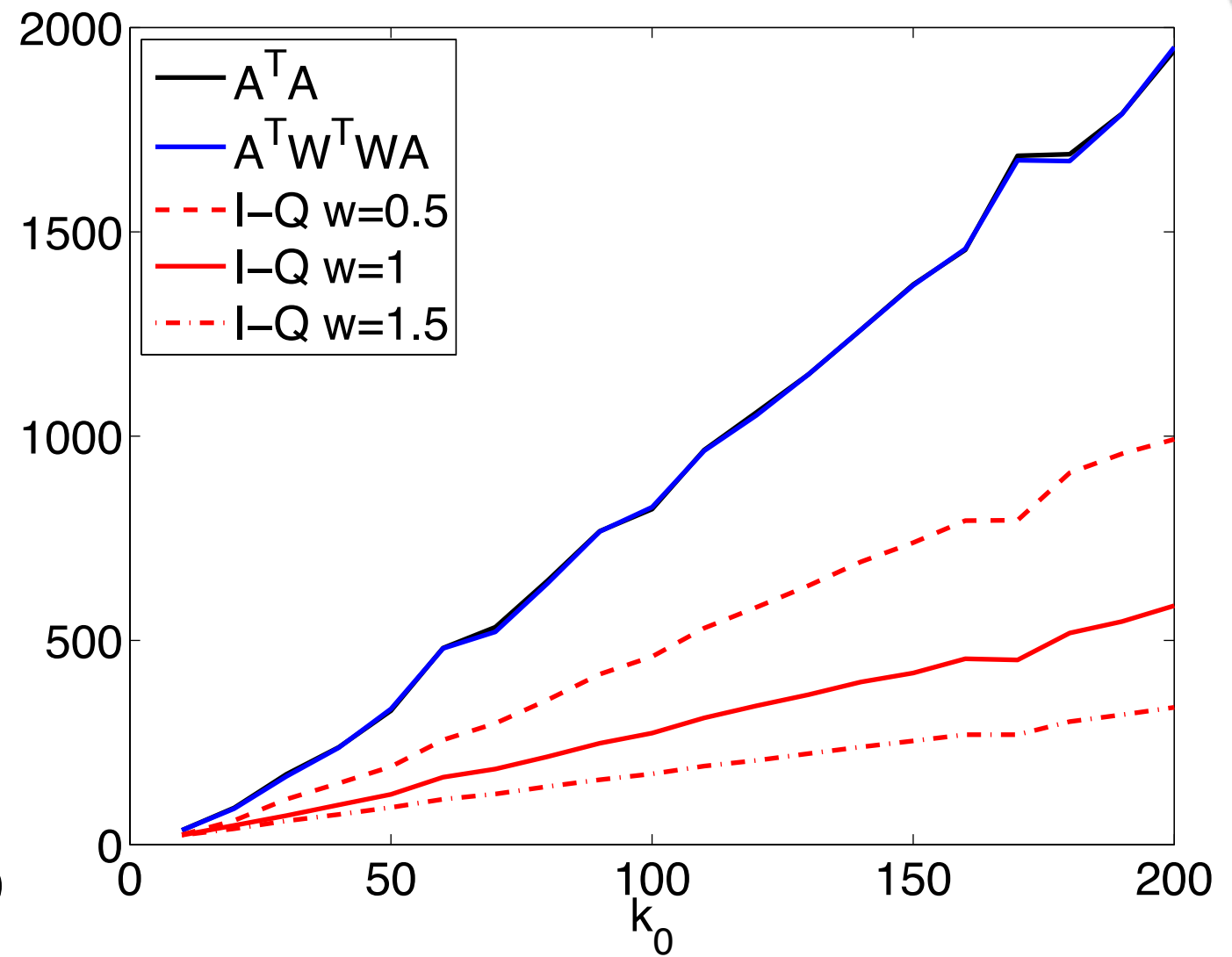


1D results

eigenvalues



of CG iterations



2D results

Marmousi, 304 x 1100, $f=20$, $h=10$

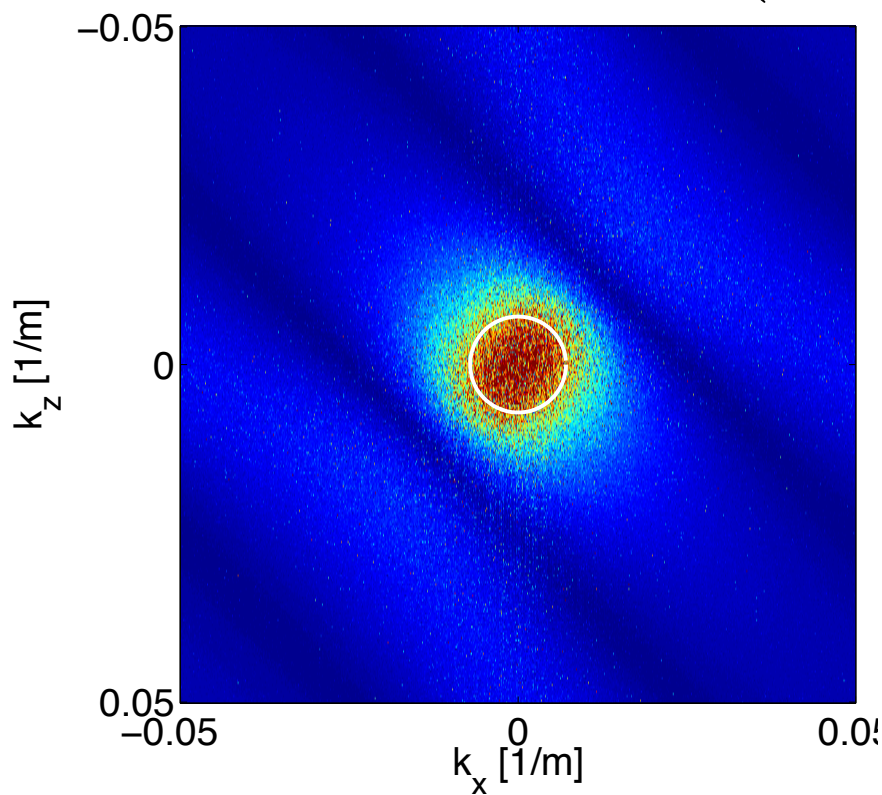
- CG + Kaczmarz (CGMN)
- BiCGstab + ILU(0)
- SQMR + ML [Bollhofer et al, '08]

2D results

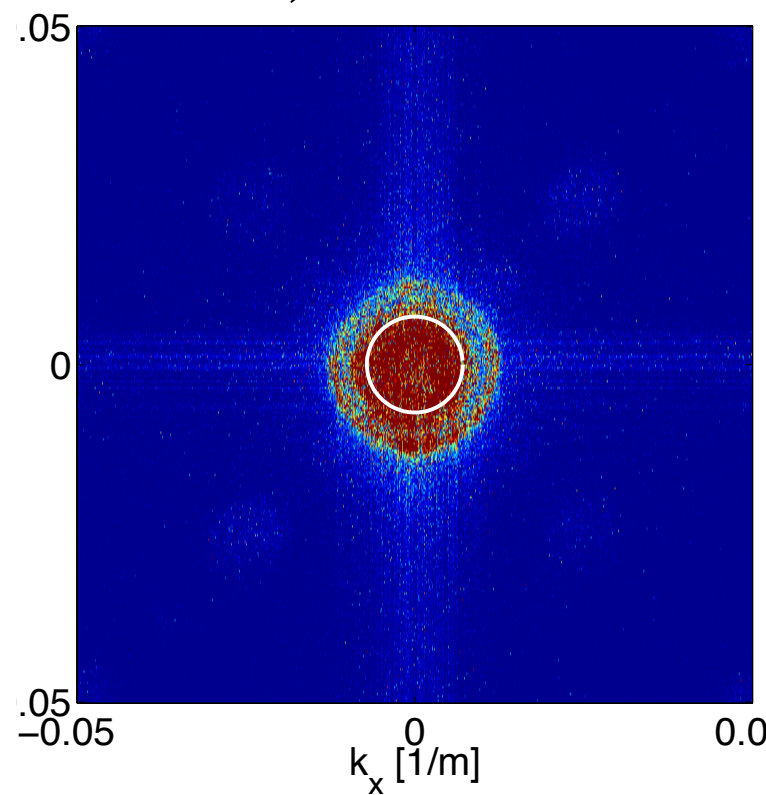
solve $A\mathbf{r} = 0$ starting from random vector

$$\mathbf{r}_1 = (I - M^{-1}A)\mathbf{r}_0$$

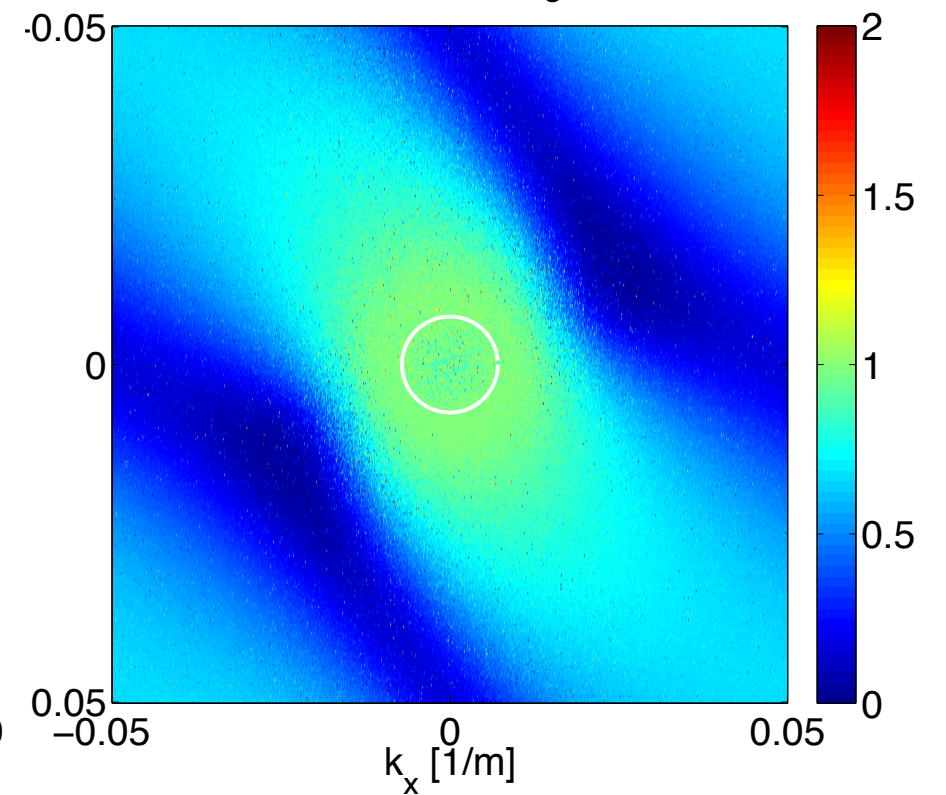
$$\mathbf{r}_1 = Q\mathbf{r}_0$$



ILU(0)



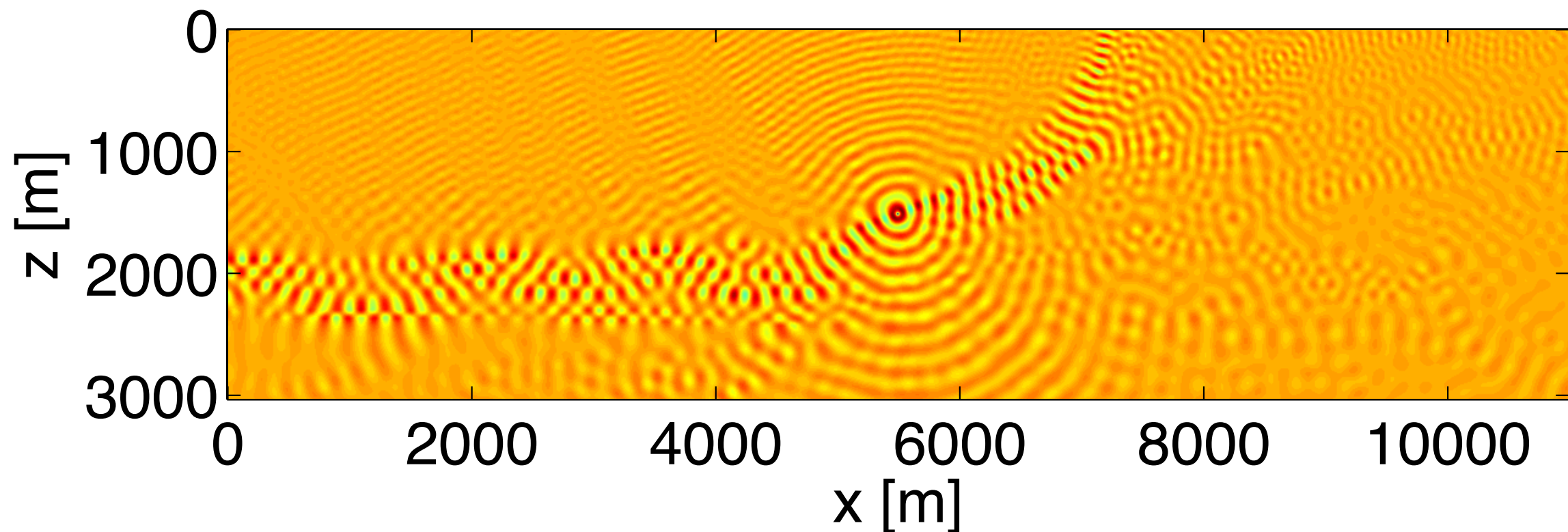
ML



Kaczmarz

2D results

	iterations	time [s]*
CG + Kaczmarz	5542	603
BiCGstab + ILU(0)	div.	div.
SQMR + ML	514	379

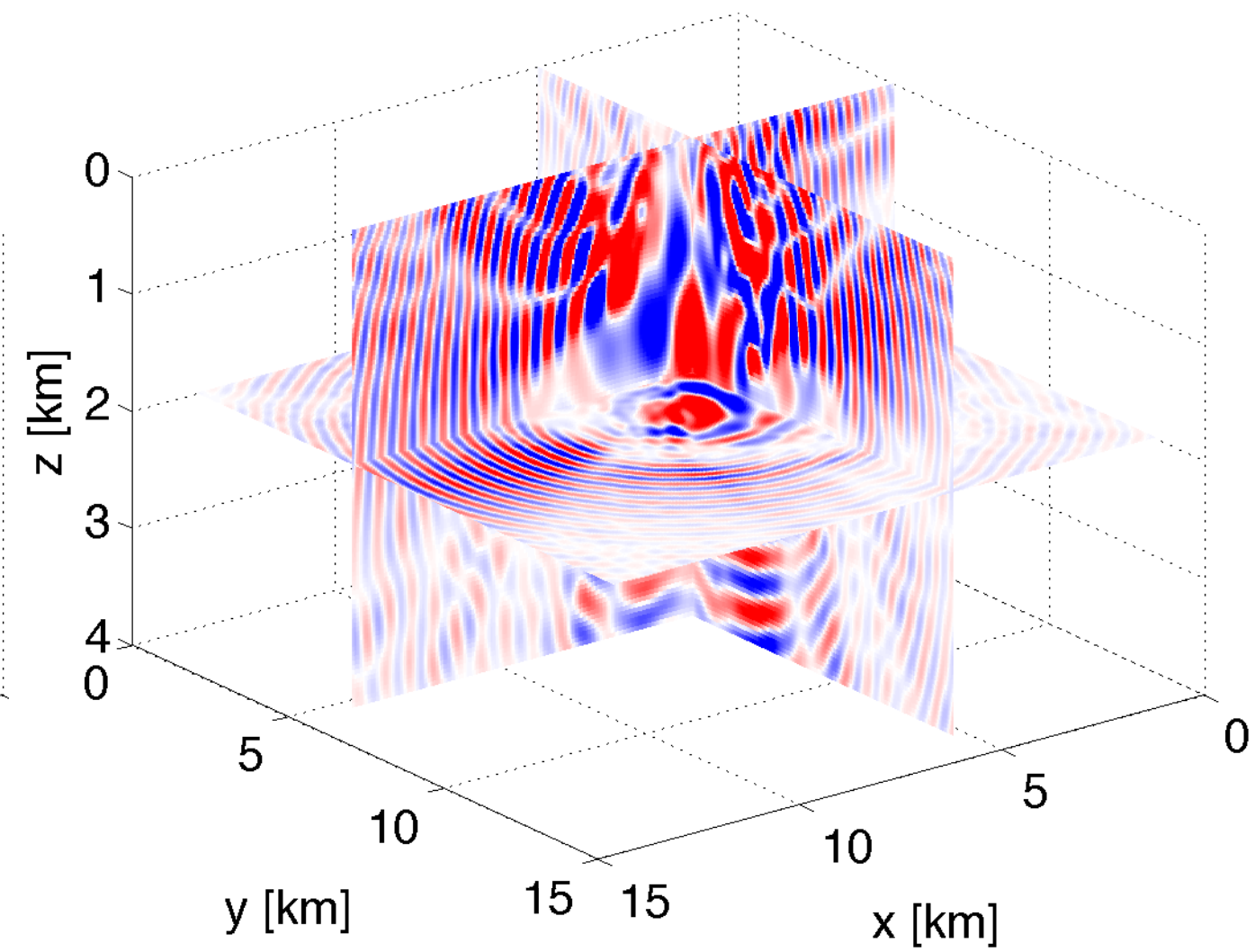
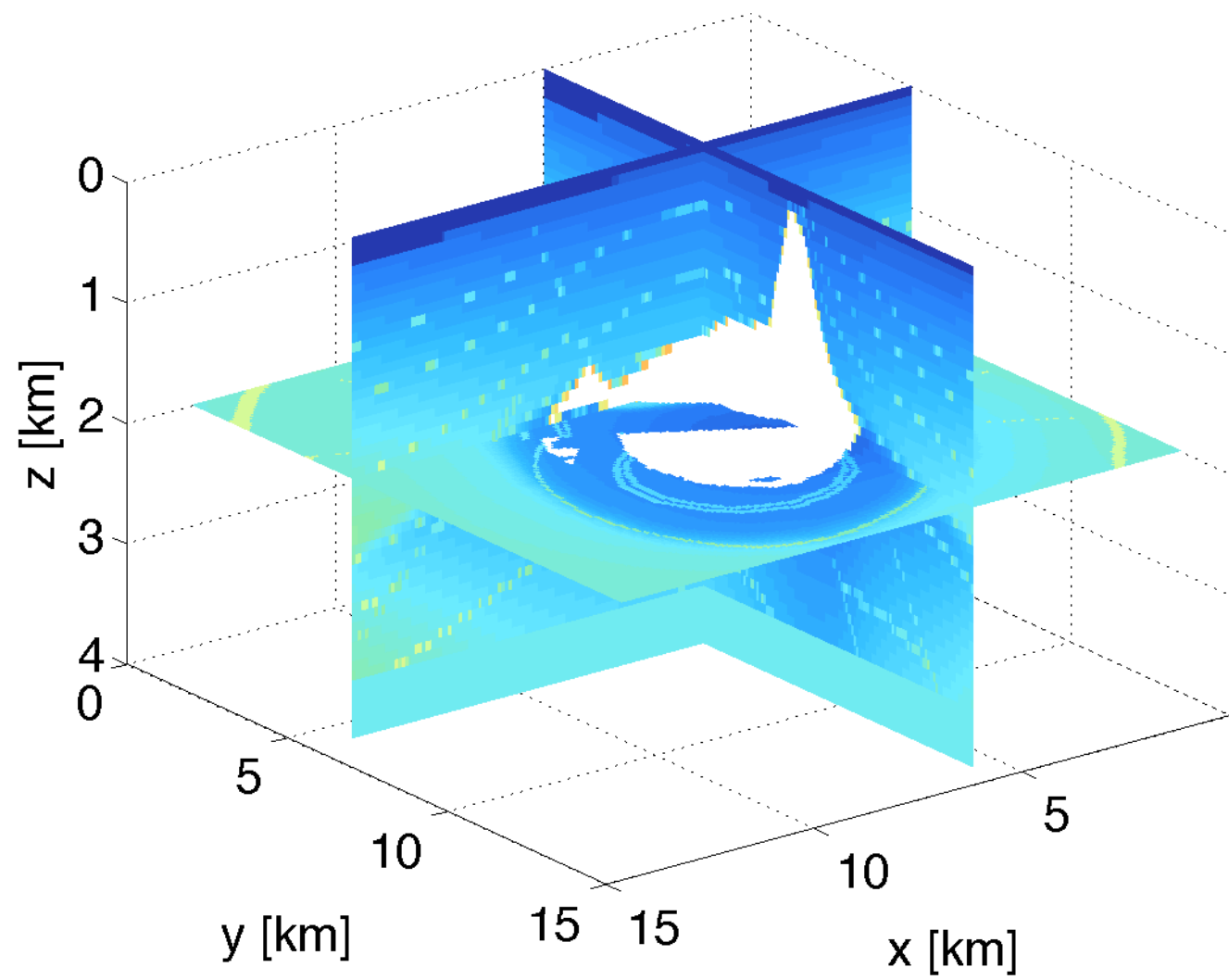


Parallelization

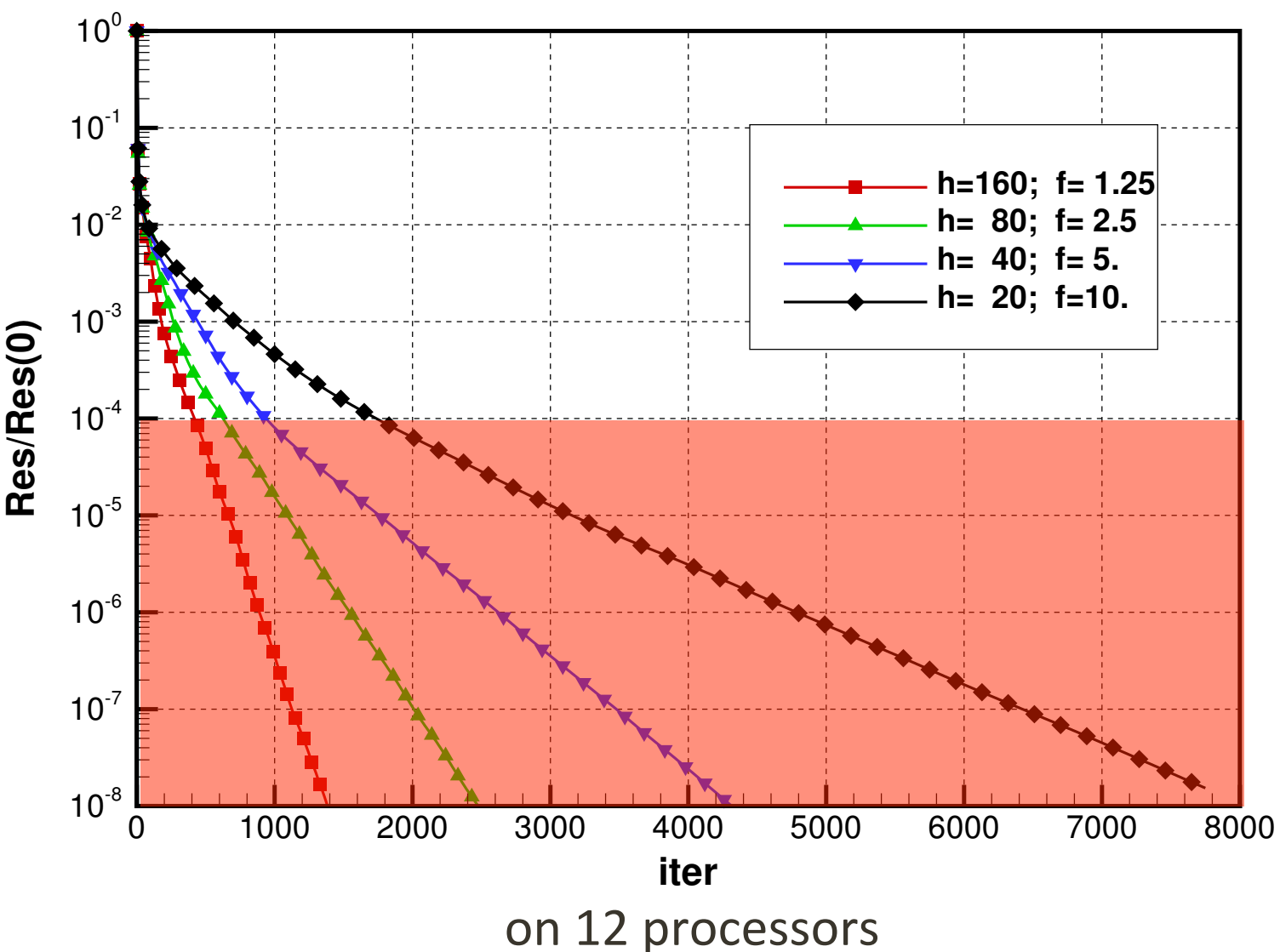
- divide domain in blocks
- Kaczmarz sweeps on blocks are done in parallel (CARP)
- average boundary elements between each sweep
- convergence guaranteed

SEG/EAGE salt

7-point stencil, ABC



SEG/EAGE salt



f	h	iterations
1.25	160	310
2.5	80	510
5	40	760
10	20	1780

on 1 processor, $\epsilon = 10^{-4}$

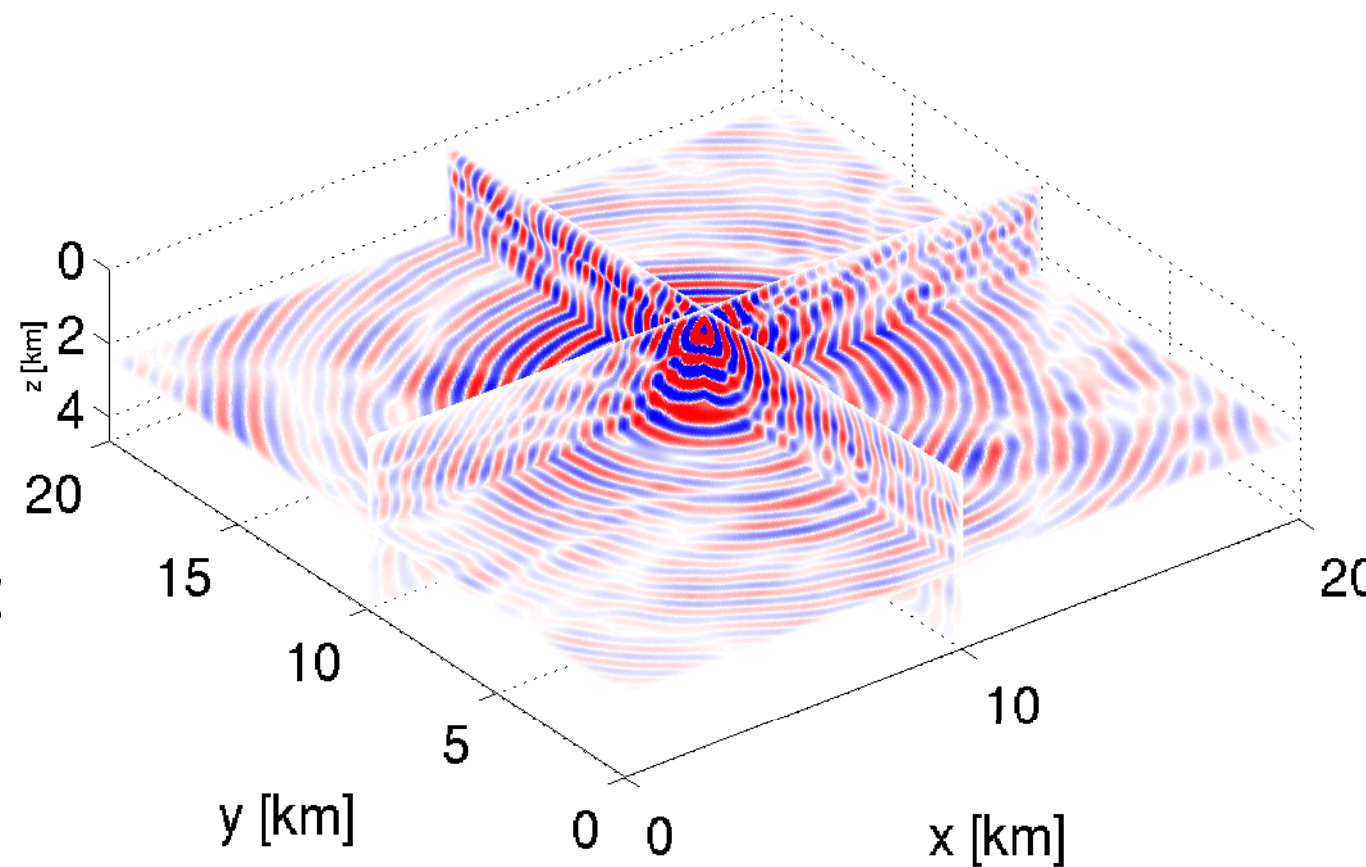
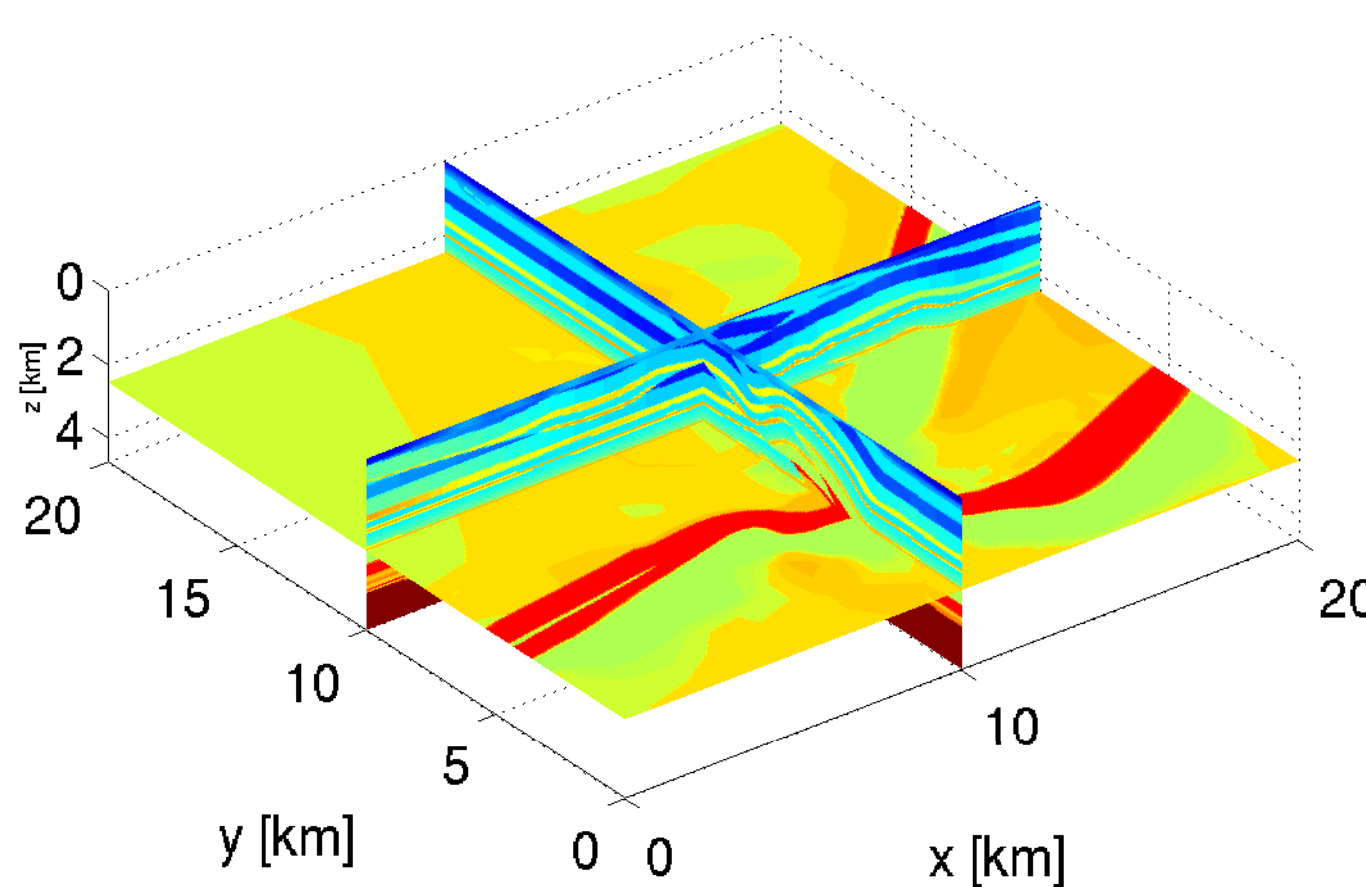
SEG/EAGE salt

grid: 105 x 338 x 338, h=40, f=5, $\epsilon = 10^{-4}$

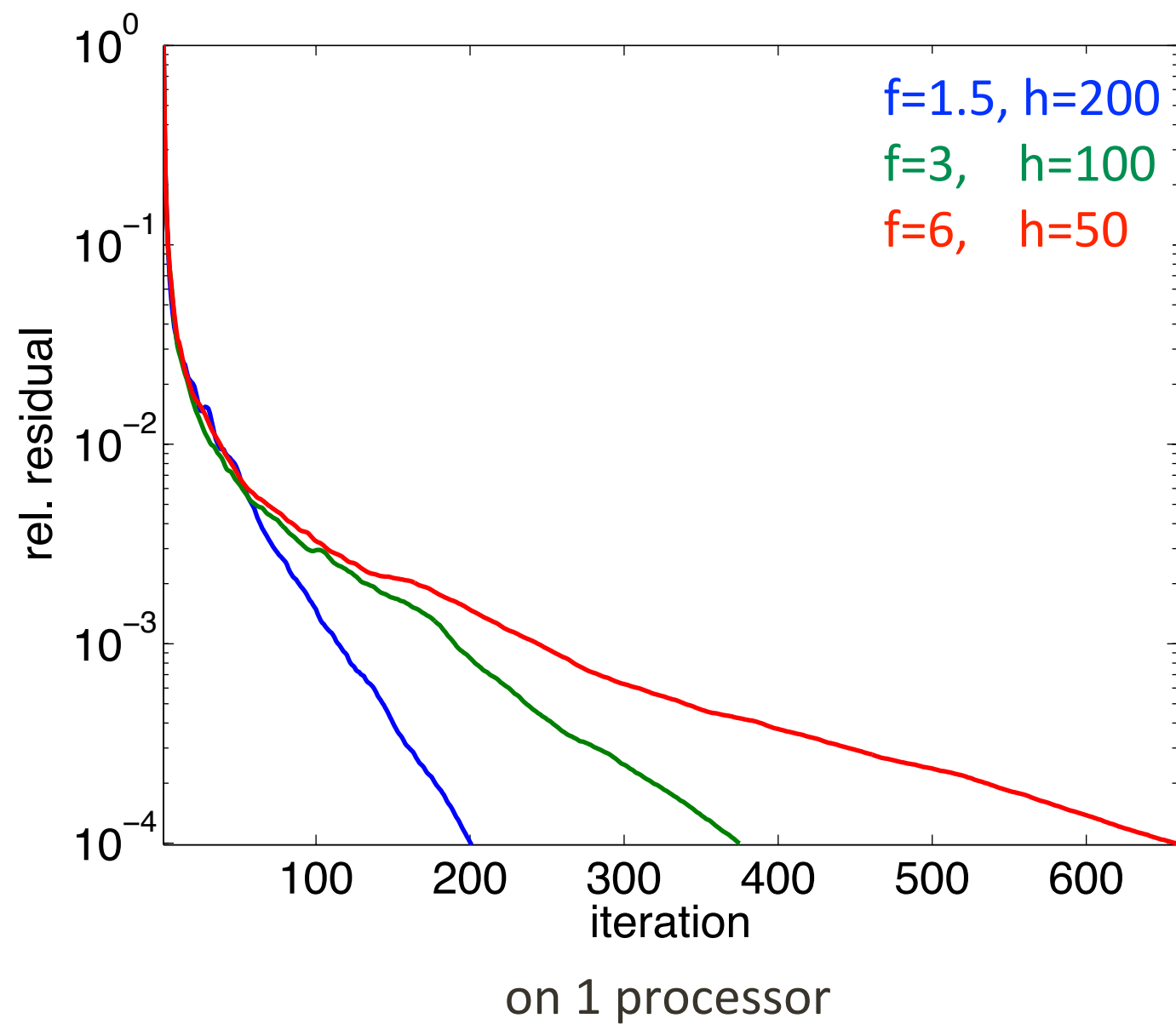
np	iter	time (s)	efficiency
1	621	4444.90	1.00
2	619	3091.10	0.72
4	593	1335.00	0.83
8	599	737.90	0.75

Overthrust

27 point stencil (2nd order), PML



Overthrust



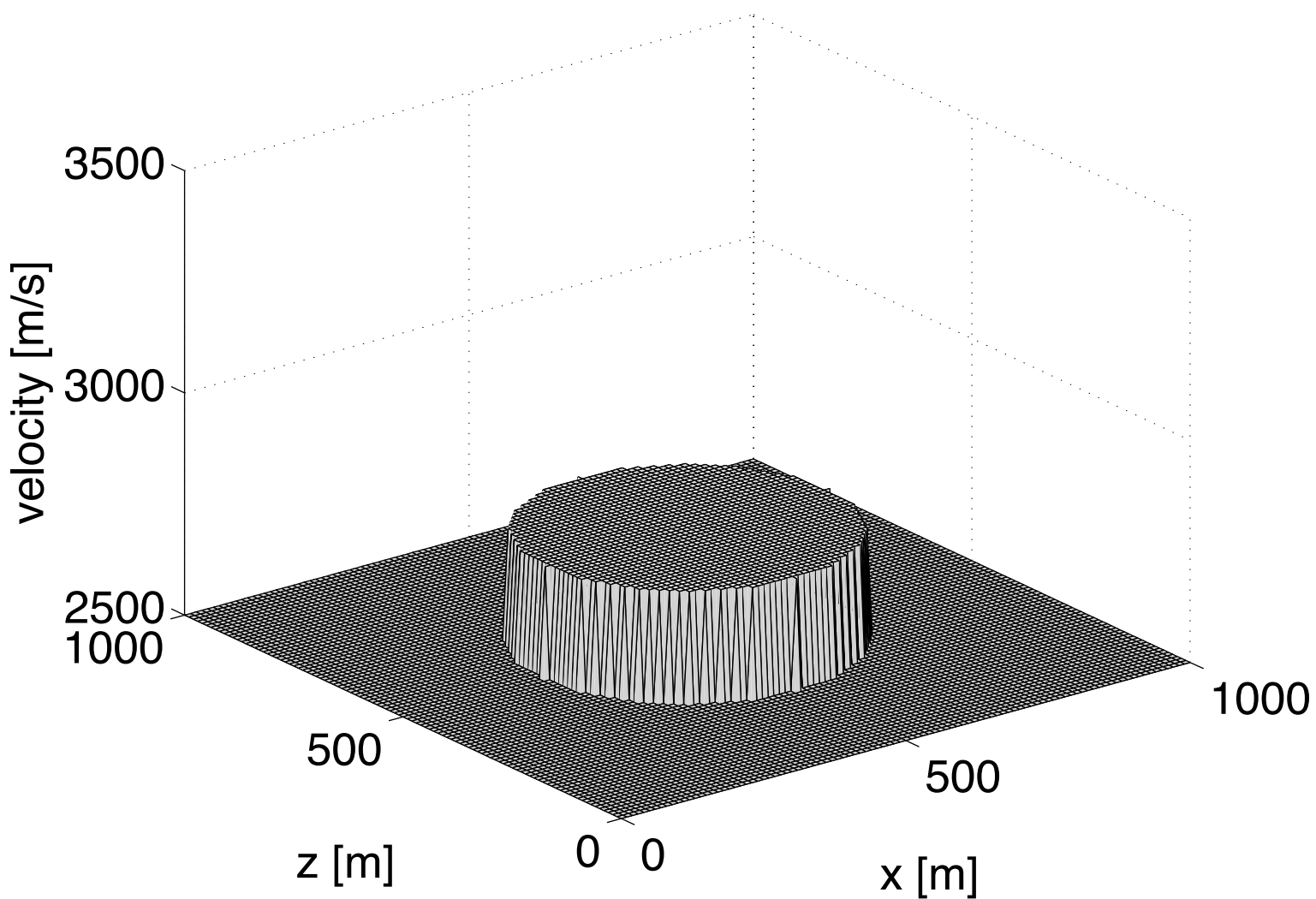
Overthrust

grid: 47x201x201, h=100, f=3 Hz, $\epsilon = 10^{-4}$

np	iter	time	efficiency
1	659	20785.40	1.00
2	657	11306.90	0.92
4	596	4882.50	0.96
8	603	3960.10	0.60

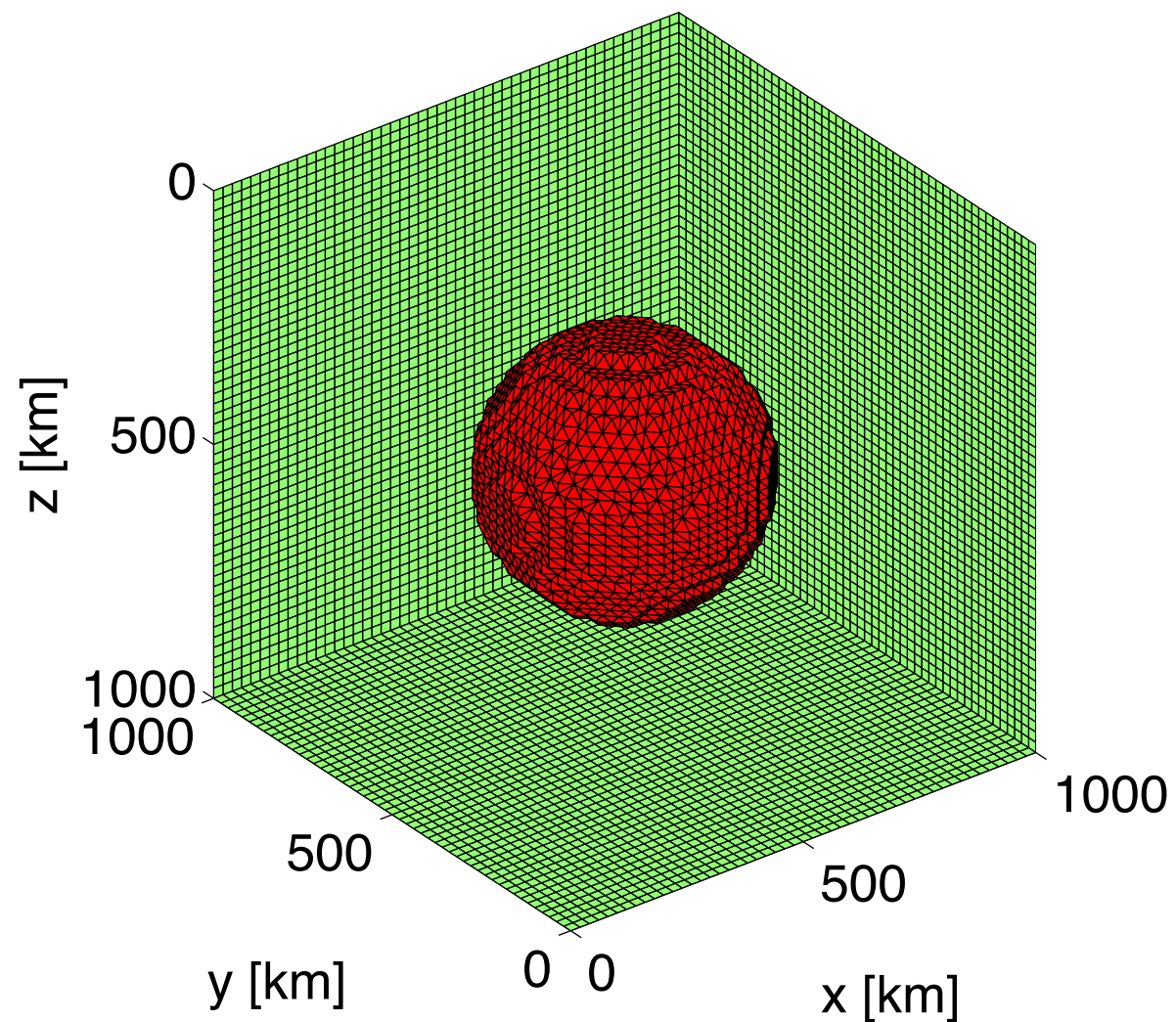
Inversion

Camembert model in 2D ...



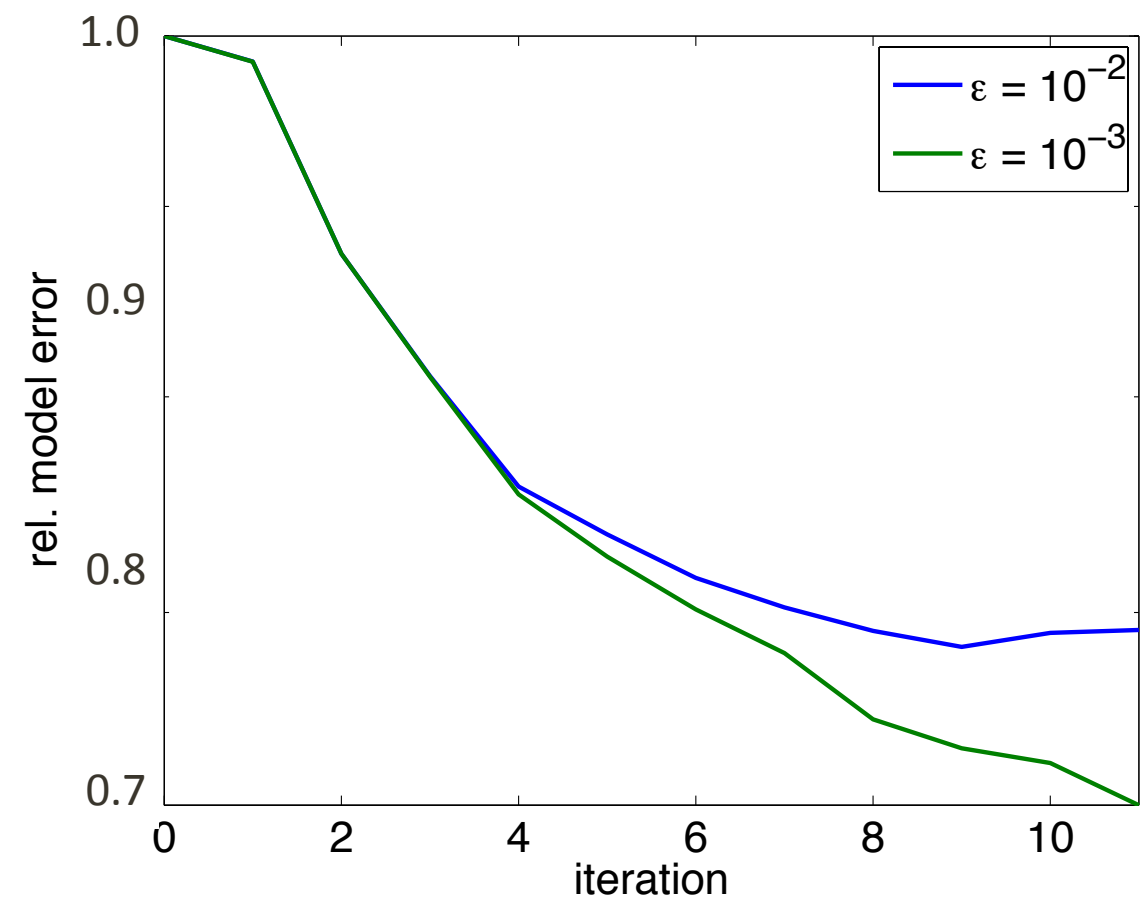
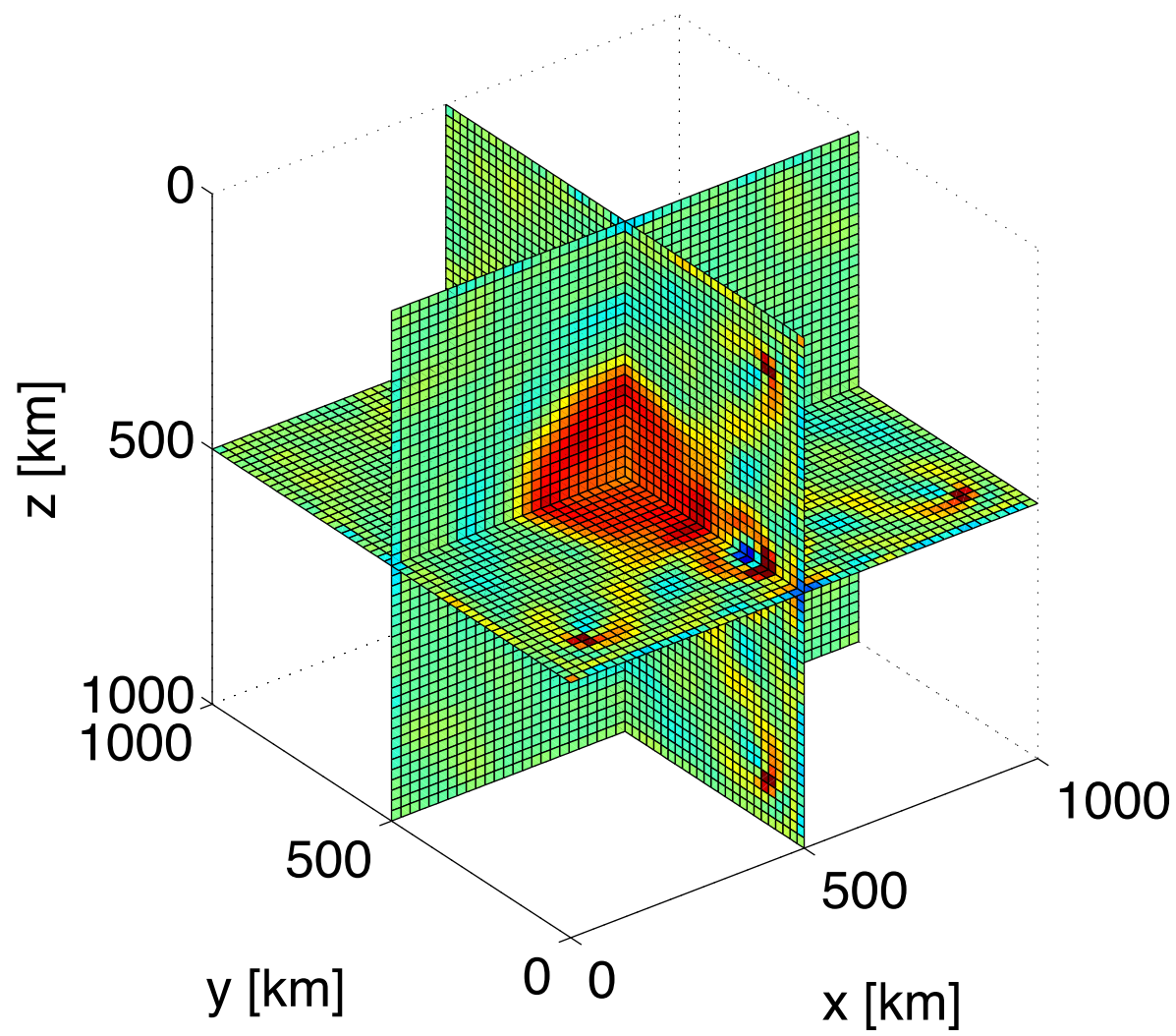
Inversion

EDAM model in 3D !



Inversion

transmission setup, 9 sources,
3 frequencies



Conclusions

- simple, robust and generic preconditioner
- no overhead, cheap to apply
- easy to parallelize

Future plans

- efficient implementation of sweeps using multi-threading
- investigate BlockCG
- incorporate in inversion
- high-order schemes



<http://cs.haifa.ac.il/~gordon/soft.html>

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