Up / down wavefield decomposition
by sparse inversion

Joost van der Neut (Delft University of Technology)
Felix Herrmann (University of British Columbia)
The forward problem

Flux-normalized composition matrix (Wapenaar, 1998):

$$\hat{\mathbf{L}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\omega Q)^{-\frac{1}{2}} \hat{\mathcal{H}}_1^{-\frac{1}{2}} & (\omega Q)^{\frac{1}{2}} \hat{\mathcal{H}}_1^{-\frac{1}{2}} \\
(\omega Q)^{-\frac{1}{2}} \hat{\mathcal{H}}_1^{\frac{1}{2}} & -(\omega Q)^{\frac{1}{2}} \hat{\mathcal{H}}_1^{\frac{1}{2}}
\end{pmatrix}$$

Square-root of the Helmholtz operator:

$$\hat{\mathcal{H}}_1 \hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2$$

$$\hat{\mathcal{H}}_2 = \left(\frac{\omega}{c}\right)^2 + \mathcal{Q} \frac{\partial}{\partial x} \left( \frac{1}{\mathcal{Q}} \frac{\partial}{\partial x} \cdot \right)$$

Spatial derivatives along receiver array
Numerical computation of the square-root operator

Computation of the Helmholtz operator at each frequency:

\[ \hat{H}_2 = \left( \frac{\omega}{c} \right)^2 + \frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial}{\partial x} \right). \]

Eigenvalue decomposition (Grimbergen et al., 1998):

\[ \hat{H}_2^{1/n} = \hat{L} \Lambda^{1/n} \hat{L}^{-1} \]

n-th root
Flux-normalized inverse composition matrix (Wapenaar, 1998):

\[
\hat{\mathbf{L}}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\hat{\mathcal{H}}_1^{\frac{1}{2}} (\omega Q)^{-\frac{1}{2}} & \hat{\mathcal{H}}_1^{-\frac{1}{2}} (\omega Q)^{\frac{1}{2}} \\
\hat{\mathcal{H}}_1^{\frac{1}{2}} (\omega Q)^{-\frac{1}{2}} & -\hat{\mathcal{H}}_1^{-\frac{1}{2}} (\omega Q)^{\frac{1}{2}}
\end{pmatrix}
\]

Square-root of the Helmholtz operator:

\[
\hat{\mathcal{H}}_1 \hat{\mathcal{H}}_1 = \hat{\mathcal{H}}_2
\]

\[
\hat{\mathcal{H}}_2 = \left( \frac{\omega}{c} \right)^2 + Q \frac{\partial}{\partial x} \left( \frac{1}{Q} \frac{\partial}{\partial x} \right)
\]
Problems

1. Noise

\[
\begin{pmatrix}
\hat{p}^+ \\
\hat{p}^-
\end{pmatrix} = \hat{L}^{-1} \left[ \begin{pmatrix}
\hat{p}_{\text{signal}} \\
\hat{v}_{\text{signal}}
\end{pmatrix} + \begin{pmatrix}
\hat{p}_{\text{noise}} \\
\hat{v}_{\text{noise}}
\end{pmatrix} \right]
\]

2. Singularities

In fk-domain:

\[
\hat{L}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{\frac{k_z}{\omega Q}} & \sqrt{\frac{\omega Q}{k_z}} \\
\sqrt{\frac{\omega Q}{k_z}} & -\sqrt{\frac{k_z}{\omega Q}}
\end{pmatrix}
\]

Vertical wavenumber:

\[ k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2} \]

At critical angles:

\[ \frac{\omega^2}{c^2} = k_x^2 \]

Numerical instability
Sparsity promotion


– Applications in geophysics: Herrmann et al. (2008)

– The solution is assumed to be sparse in a transform domain:

\[
\begin{align*}
\mathbf{p}^\pm &= \mathbf{S}^* \mathbf{x}^\pm \\
\mathbf{S} &= \mathbf{C}_2 \otimes \mathbf{W}
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad \left\| \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} \right\|_1 \\
\text{subject to} & \quad \left\| \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} - L \begin{pmatrix} \mathbf{S}^* & 0 \\ 0 & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{pmatrix} \right\|_2 \leq \sigma
\end{align*}
\]

– spgl1 (Van den Berg and Friedlander, 2008)
Example

Velocity model:

- 128 sources
- 128 receivers
- Noise added (SNR=5)
Pressure data

Common Receiver Gather

- **Dowgoing field** (not changing polarity)
- **Upgoing field** (changing polarity)

Common Source Gather

FK spectrum
Particle velocity data

Common Receiver Gather

- Downgoing field (not changing polarity)
- Upgoing field (changing polarity)

Common Source Gather

FK spectrum

Time (s)

Source location (km)

Time (s)

Receiver location (km)

Frequency (Hz)

Wavenumber (1/m)
Upgoing field retrieved by exact inversion

- Results are noisy
- Instabilities at singularities
Upgoing field retrieved by exact inversion

Common Receiver Gather

– After FK-filtering

Common Source Gather

FK spectrum
Upgoing field retrieved by sparse inversion

Common Receiver Gather

- Result less noisy
- No instabilities at singularities

Common Source Gather

FK spectrum
Upgoing field retrieved without noise

– Detailed information can be lost if noise level $\sigma$ is set too high
– Instabilities can occur if noise level $\sigma$ is set too low
Extension to elastic media

- Composition matrix – Wapenaar et al. (2008)

- Potential application: OBC (Ocean Bottom Cable) decomposition (Schalkwijk et al., 2003)
Application: Up / down deconvolution

- OBC data (Amundsen, 2001) → Free-surface multiple elimination
- Streamer data (Majdanski et al., 2011) → Free-surface multiple elimination
- Downhole receivers (Wapenaar et al., 2008) → Interferometric redatuming / virtual source method
Conclusion

* Instabilities caused by singularities of the square-root operator and noise are avoided by sparse inversion

* Applications: dual-sensor streamer data, OBC data, downhole receivers (virtual source method)
Acknowledgements & References

- We would like to thank Tristan van Leeuwen, Tim Lin, Ning Tu, Xiang Li and Ian Hanlon (University of British Columbia) for helping with implementations and for discussions.

- This work was supported by the Dutch Technology Foundation STW (STW), applied science division of NWO and the Technology Program of the Ministry of Economic Affairs (grant DCB.7913).


- Donoho, D., 2006, For most large underdetermined systems of linear equations the minimal l(1)-norm solution is also the sparsest solution: Communications on Pure and Applied Mathematics, 59, 797–829.


