# Full-waveform inversion with randomized L1 recovery for the model updates

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# Motivation

*Curse of dimensionality* for d>2

- *Exponentially* increasing *data volumes*
- *Helmholtz* requires *iterative* solvers to address *bandwidth*
- Computational complexity grows *linearly* with # of frequencies & RHS's
- High-resolution image

# Wish list

An *imaging* technology that

- is based on a *time-harmonic* PDE solver, which is easily *parallelizable*, and *scalable* to 3D
- does *not* require many passes through *all* data
- removes the *linearly* increasing costs of *iterative* solvers for increasing numbers of frequencies & RHS's

# Key technologies

Simultaneous sources & phase encoding [Beasley, '98, Berkhout, '08] [Morton, '98, Romero, '00]

- supershots [Krebs et.al., '09, Operto et. al., '09, Herrmann et.al., '08-10']

Stochastic optimization & machine learning [Bertsekas, '96]

- stochastic gradient descent

Compressive sensing [Candès et.al, Donoho, '06]

- *sparse* recovery & *randomized* subsampling

# Gauss-Newton

**Algorithm 1**: Gauss Newton

**Result**: Output estimate for the model $\mathbf{m}$

$\mathbf{m} \longleftarrow \mathbf{m}_0$; $k \longleftarrow 0$ ; `// initial model`

**while** not converged **do**

$\delta\mathbf{m}^k \longleftarrow \arg\min_{\delta\mathbf{m}} \frac{1}{2}\|\mathbf{D} - \mathcal{F}[\mathbf{m}^k;\mathbf{Q}] - \nabla\mathcal{F}[\mathbf{m}^k;\mathbf{Q}]\delta\mathbf{m}\|_{2,2}^2$

$\mathbf{m}^{k+1} \longleftarrow \mathbf{m}^k + \gamma^k\delta\mathbf{m}^k$ ; `// update with linesearch`

$k \longleftarrow k+1$;

**end**

Evaluation of $\nabla\mathcal{F}^H[\mathbf{m};\mathbf{Q}]$ and $\nabla\mathcal{F}[\mathbf{m};\mathbf{Q}]$ each require ***two*** PDE solves for *each* source & *angular* frequency

Involves inversion of a ***tall*** linear system of equations

# Dimensionality reduction

Migration operator and its adjoint requires 2 PDE solves

- use *linearity* of the *source* to turn *sequential* sources into *simultaneous* sources
- use *fewer* simultaneous sources

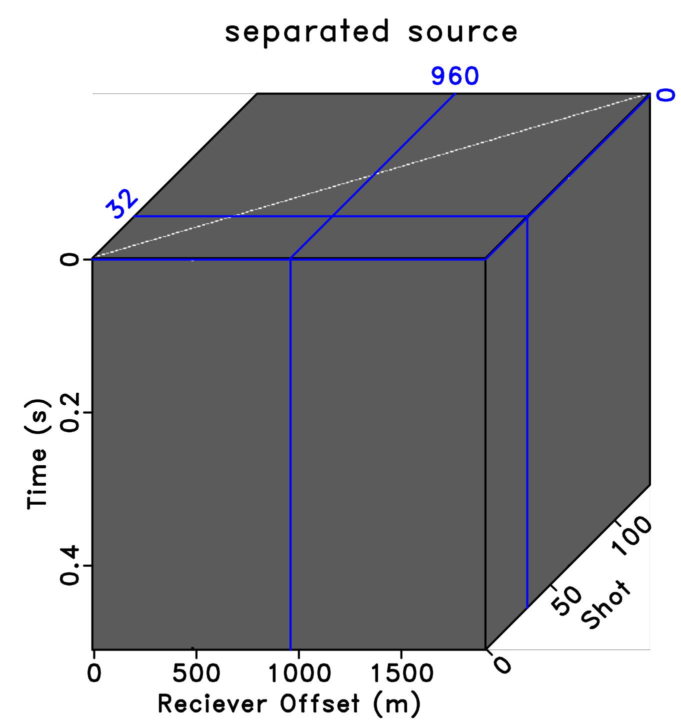
$$\delta\widetilde{\mathbf{m}} = \arg\min_{\delta\mathbf{m}} \frac{1}{2}\|\delta\underline{\mathbf{d}} - \nabla\mathcal{F}[\mathbf{m}_0;\underline{\mathbf{Q}}]\delta\mathbf{m}\|_2^2$$

$$\delta\underline{\mathbf{d}} = \text{Simultaneous-source data residue}$$

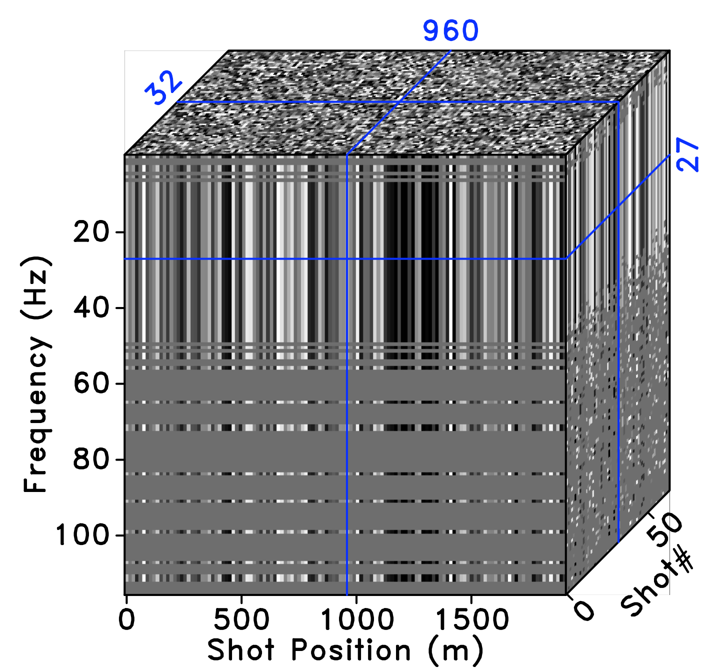
$$\underline{\mathbf{Q}} = \text{Simultaneous sources}$$

[Herrmann et. al. '08-'10]

# Supershot




$\mathbf{Q}$

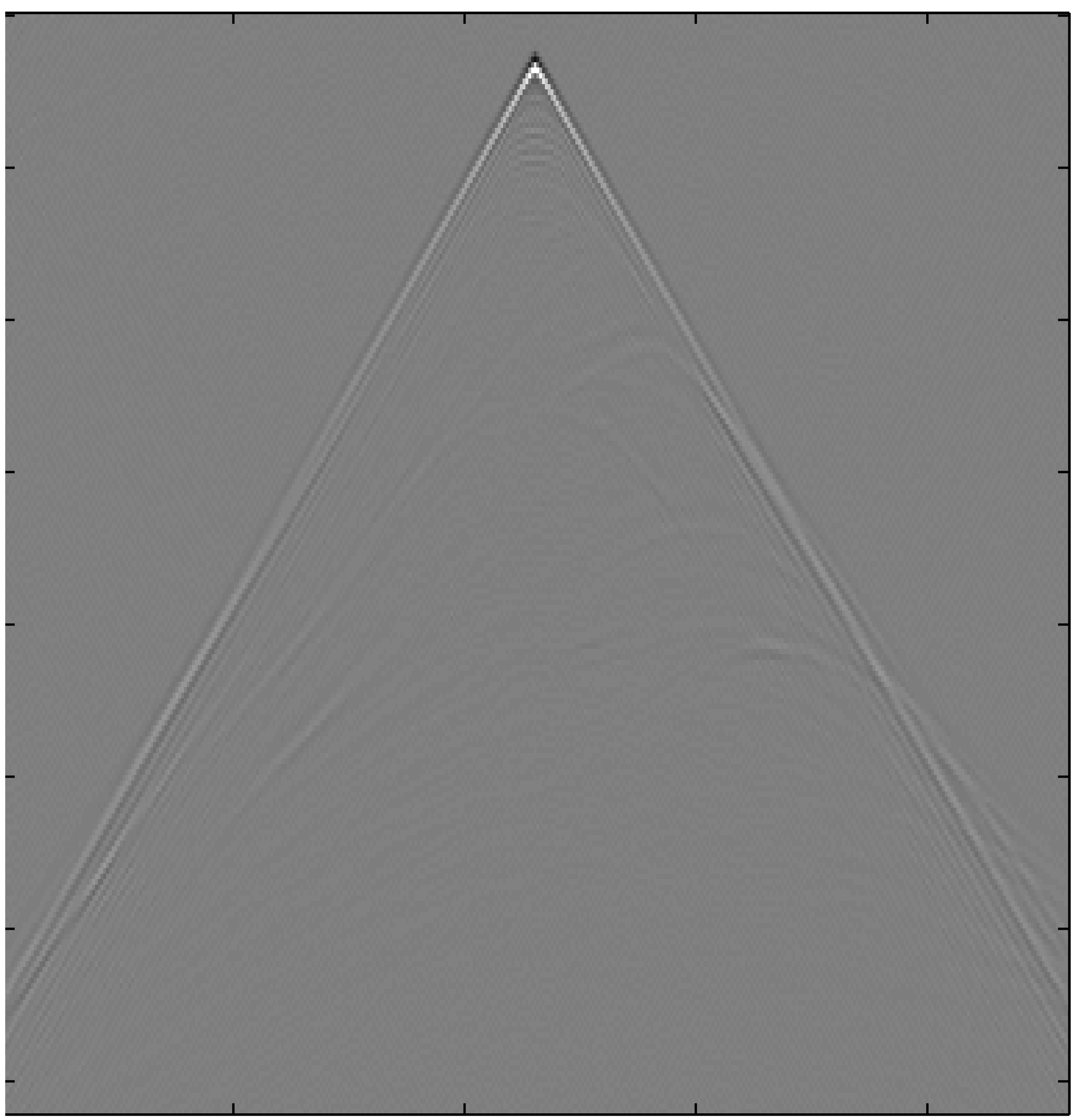



$\underline{\mathbf{Q}} = \mathbf{RMQ}$

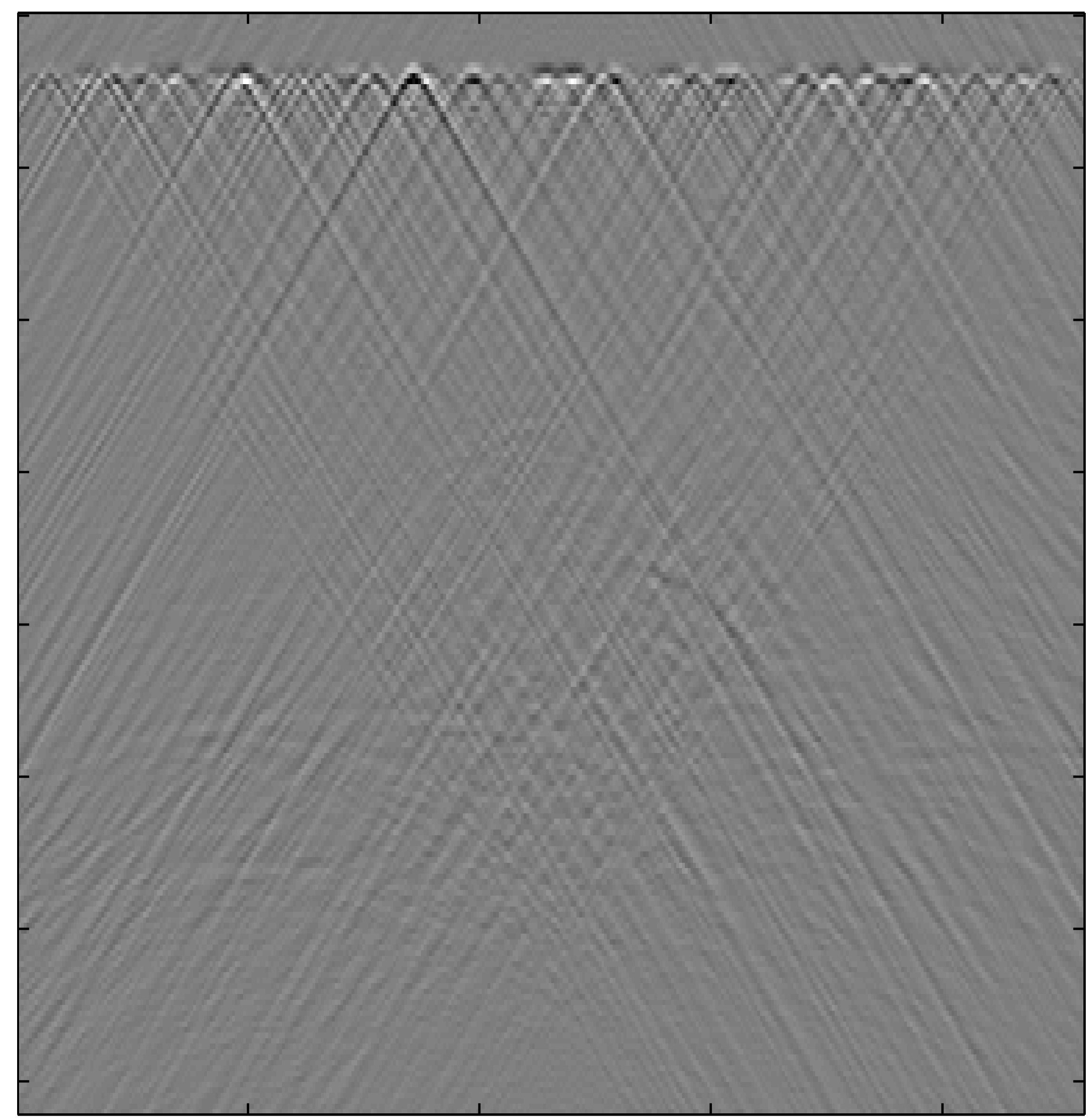
Collection of K *simultaneous-source* experiments with batch size $K \ll n_f \times n_s$

# One shot record

Sequential-Shot



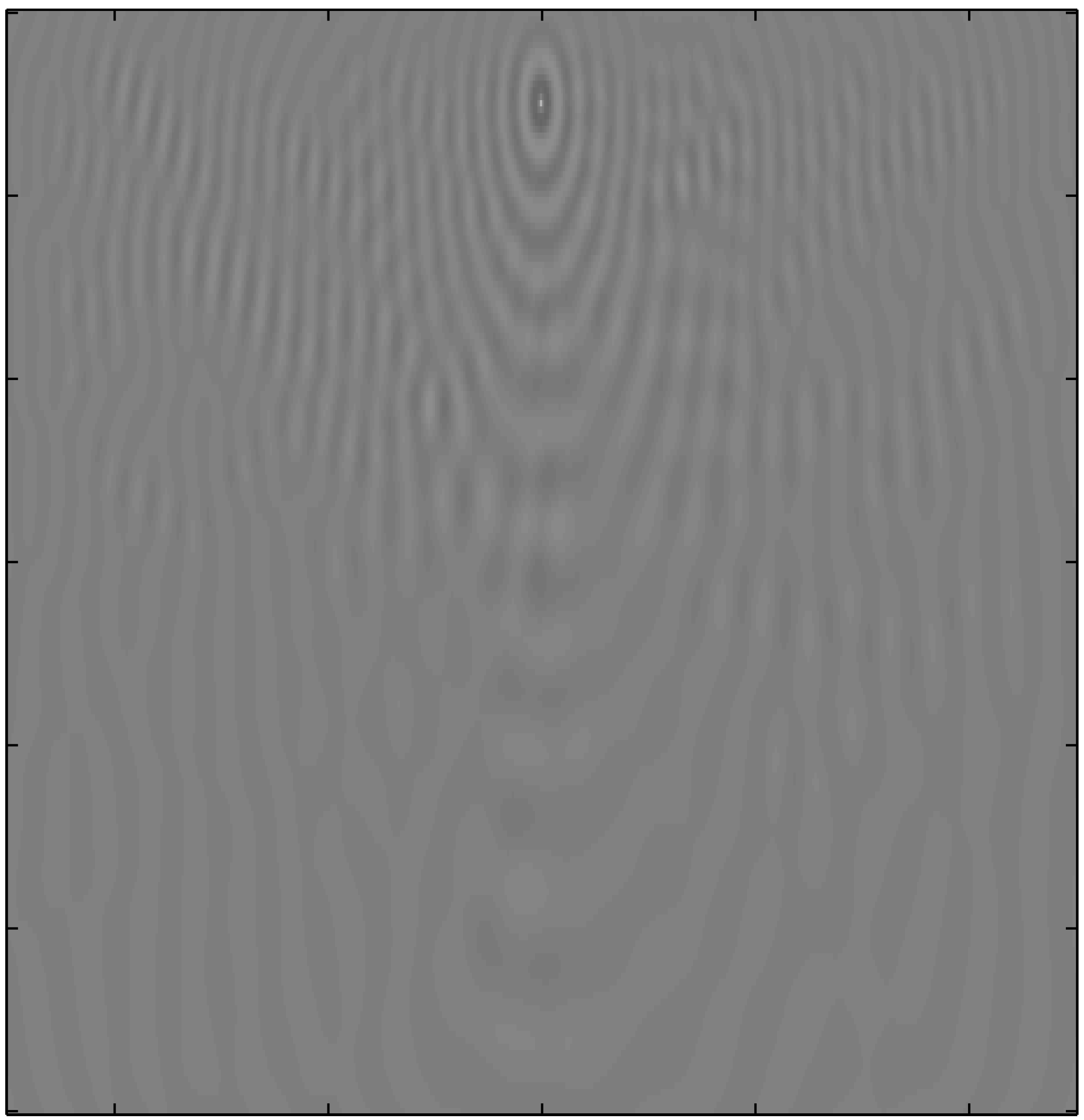
*Simultaneous*-Shot

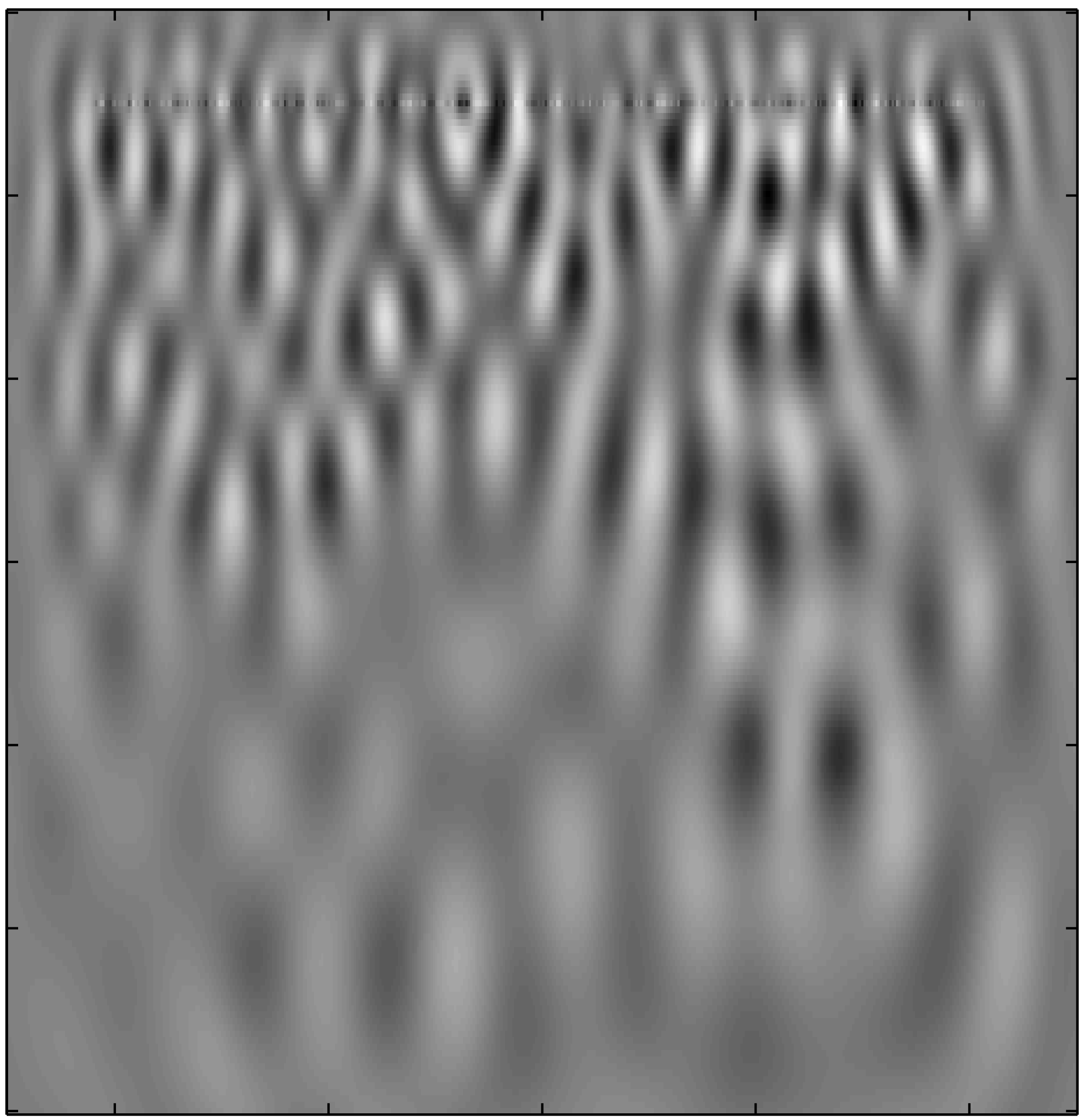


# Simultaneous shot

Sequential-source wavefield

*Simultaneous*-source wavefield





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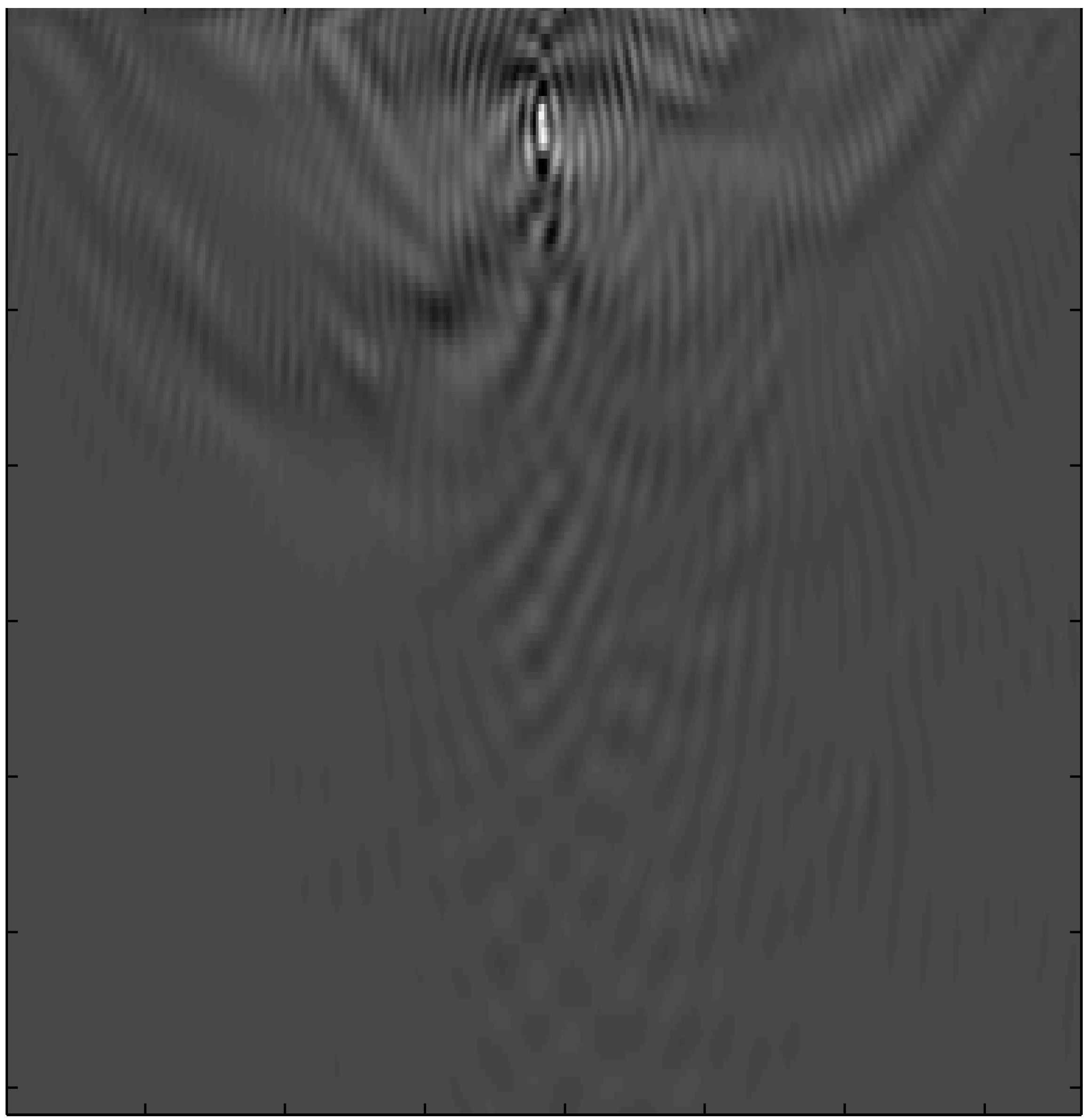
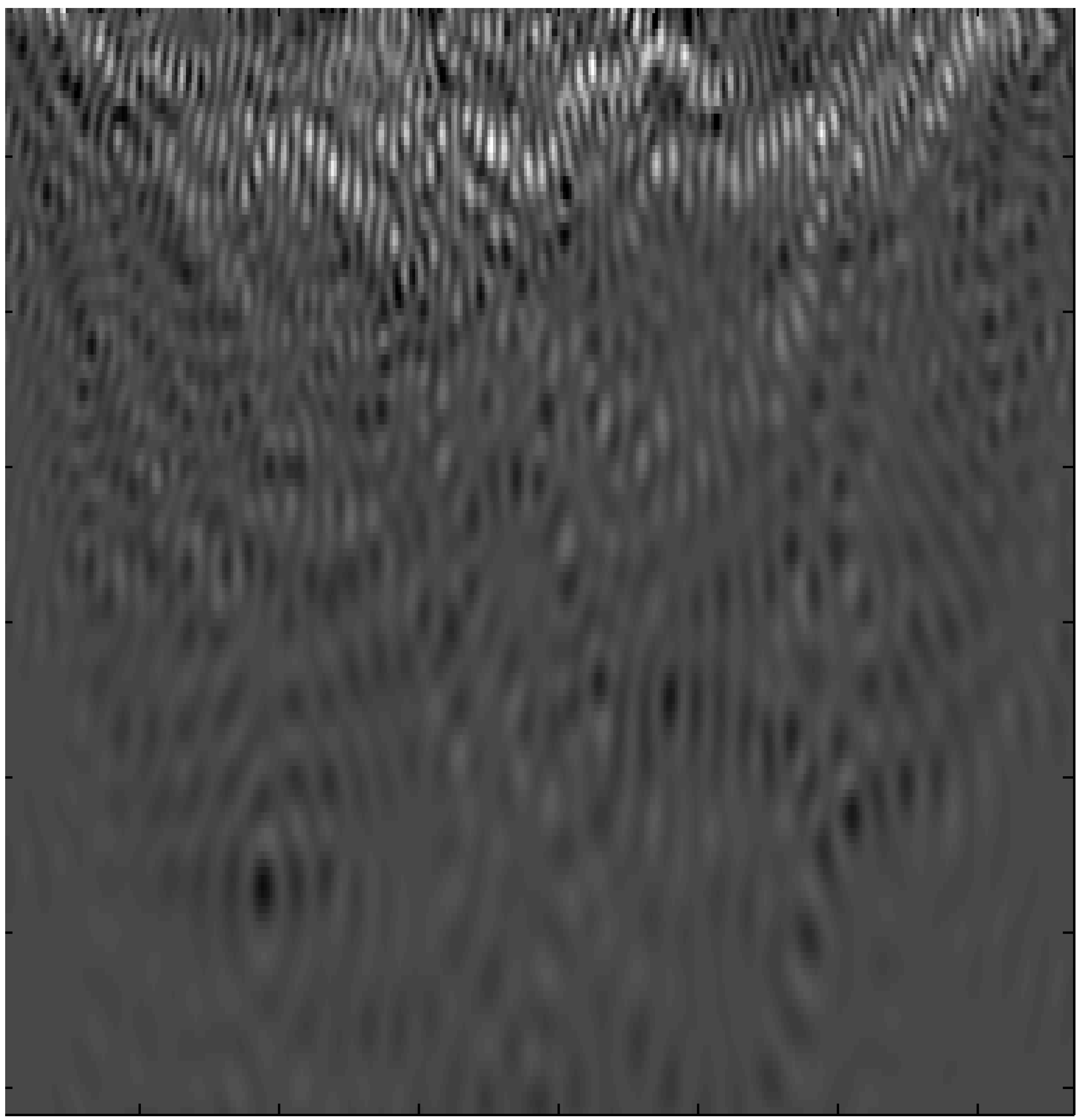
# Image from one shot

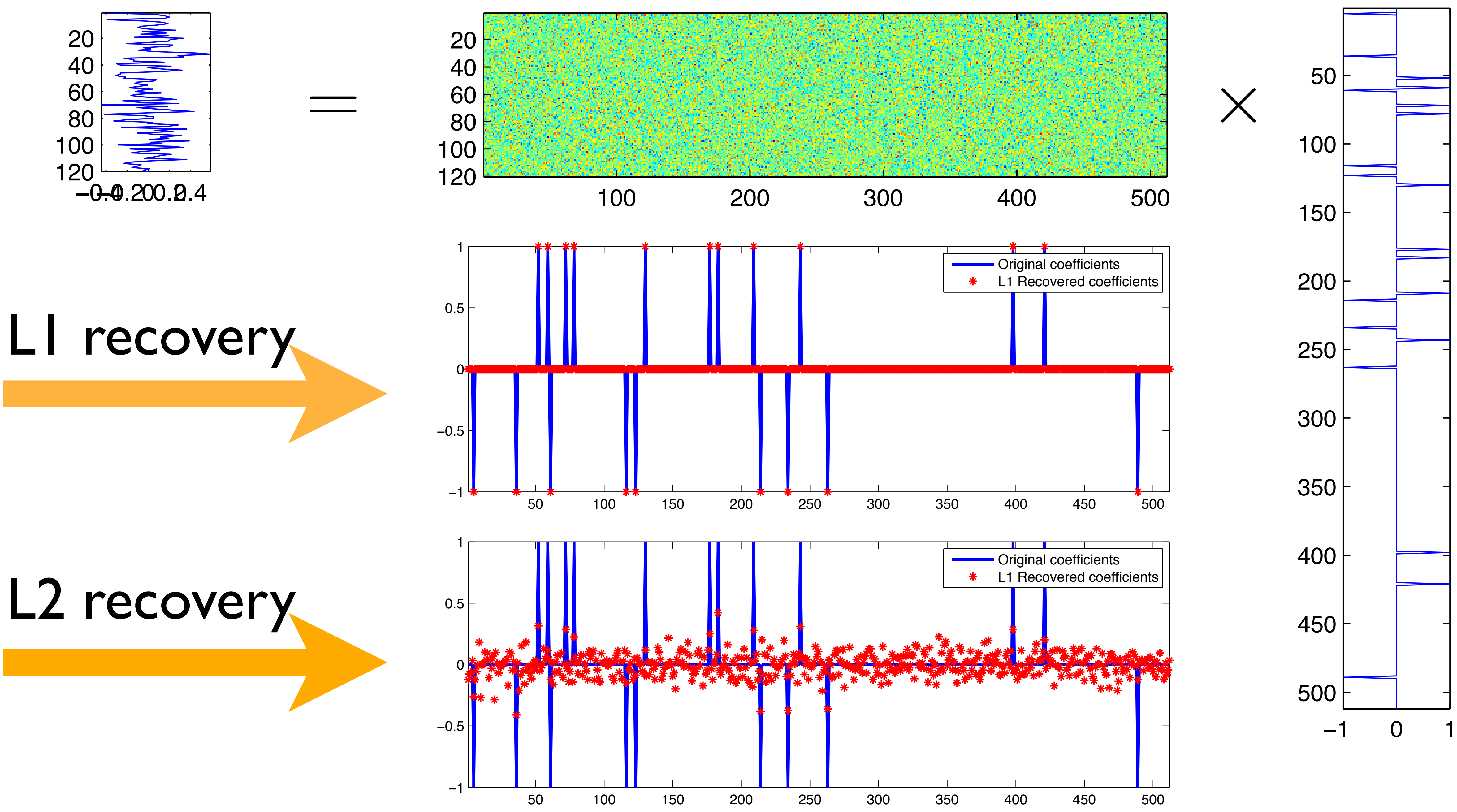
Sequential-source image



*Simultaneous*-source image

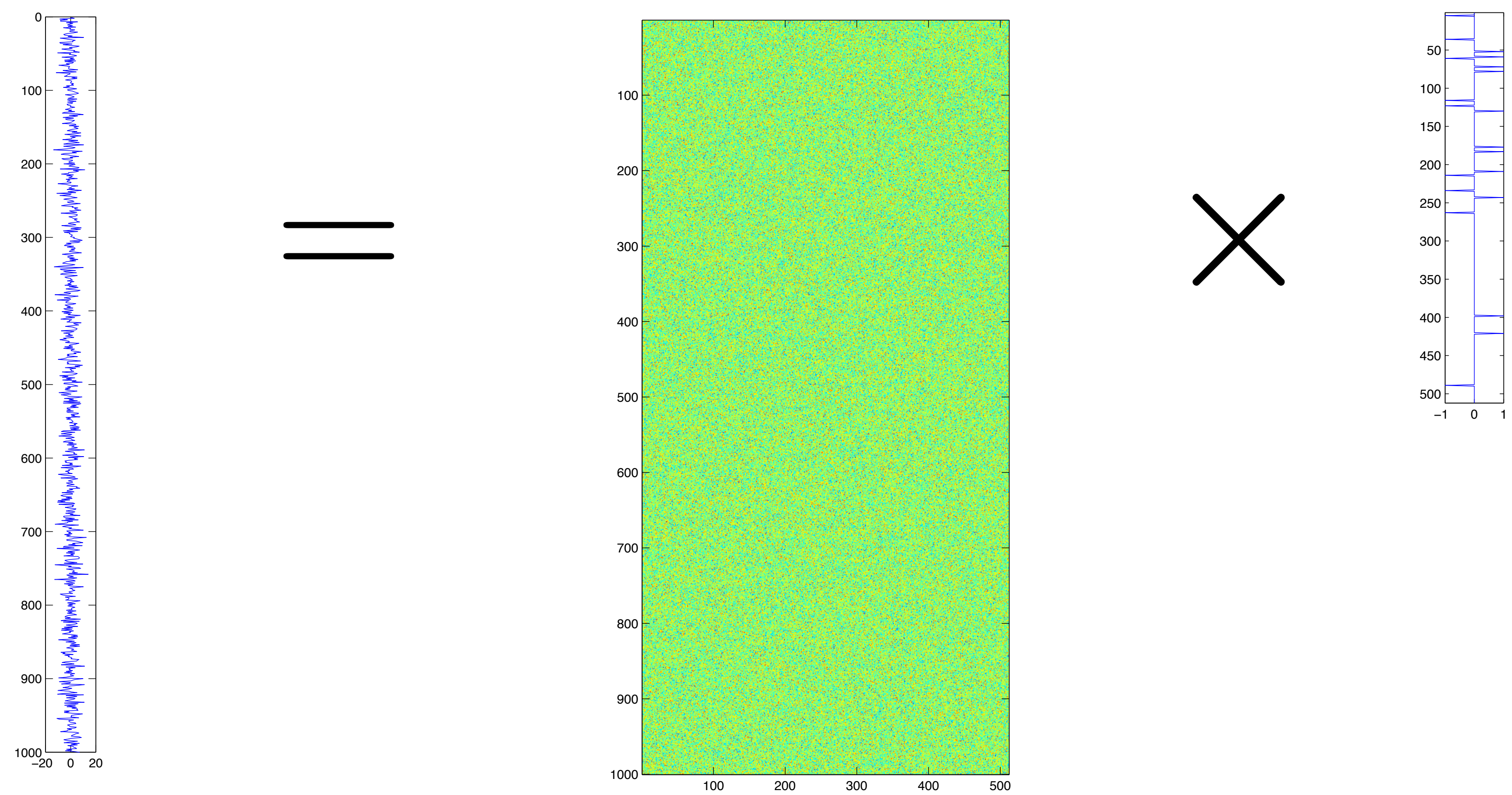


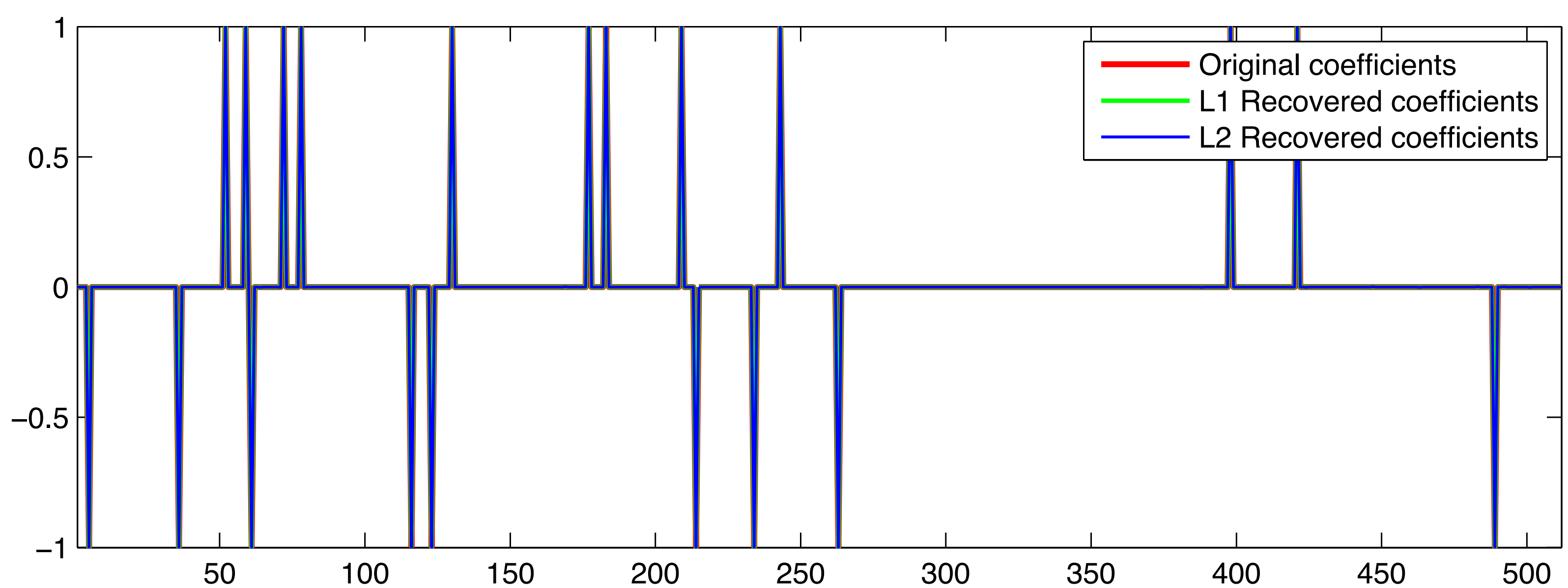
# Sparse recovery



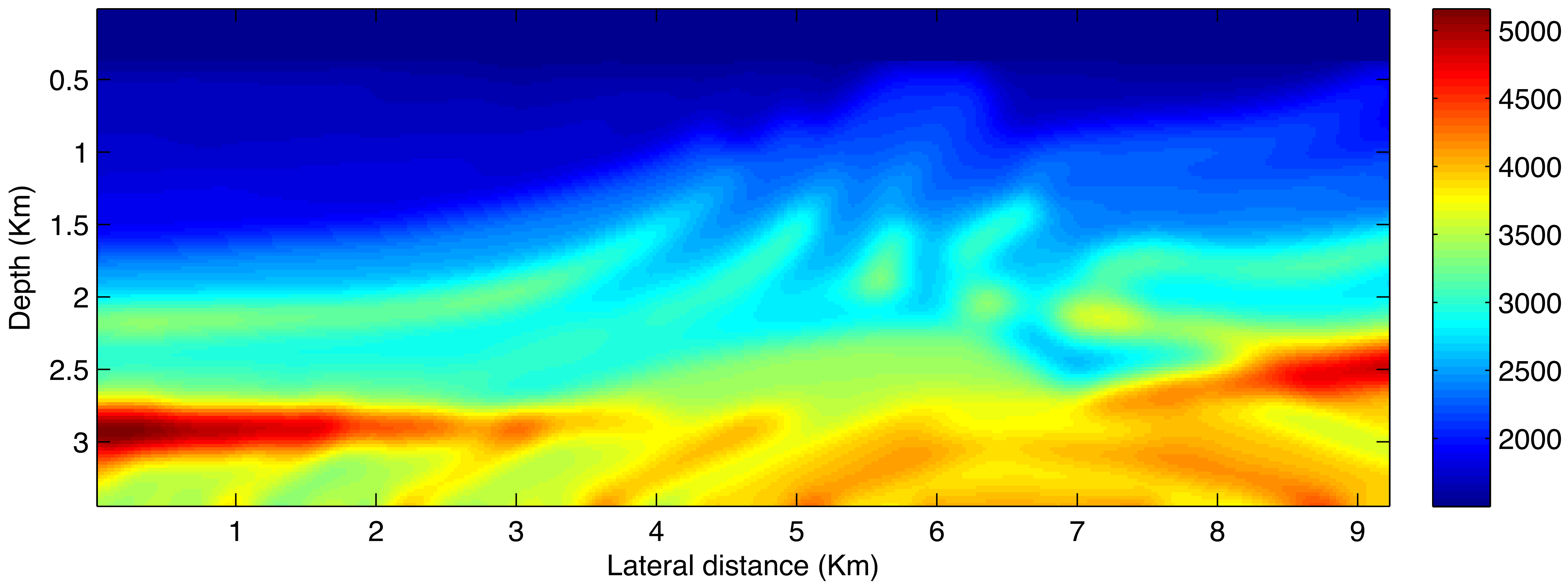

[van den Berg & Friedlander, ’08]

# Recovery

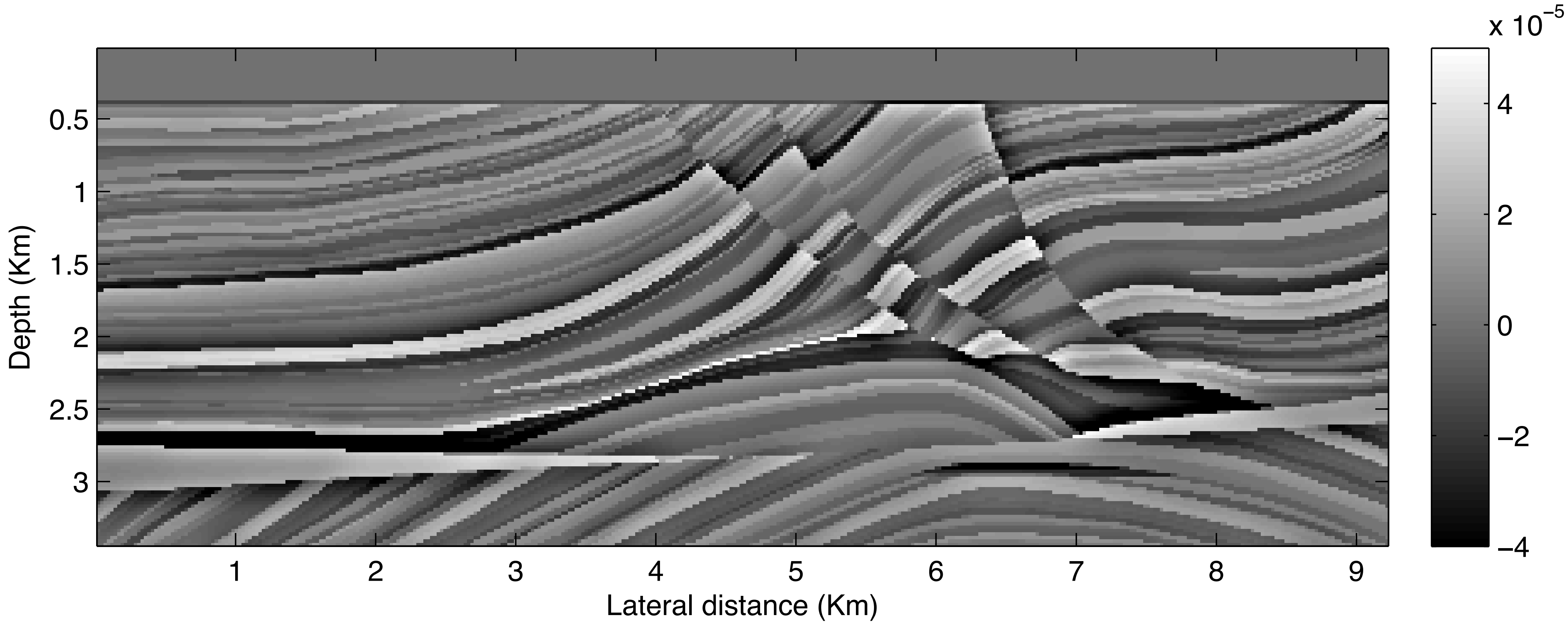



Original coefficients
L1 Recovered coefficients
L2 Recovered coefficients

# Initial model

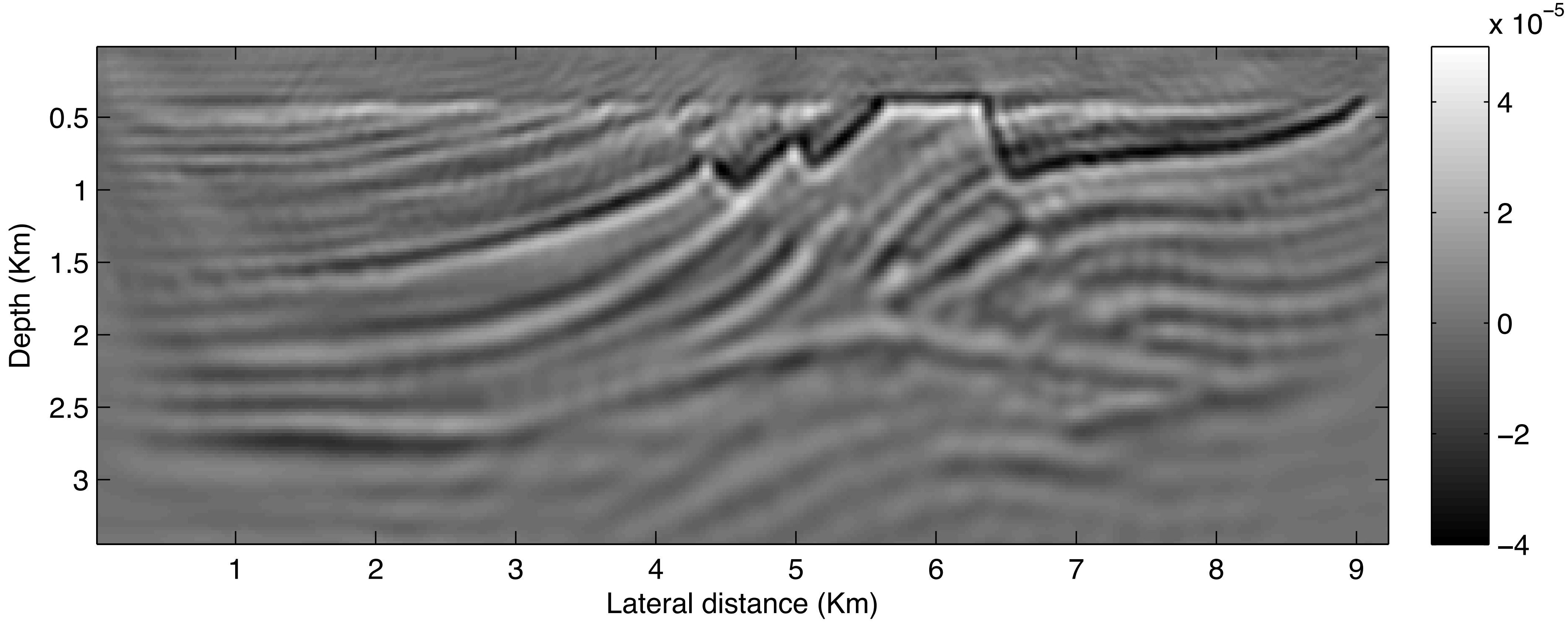


# True perturbation



# L2 for all shots

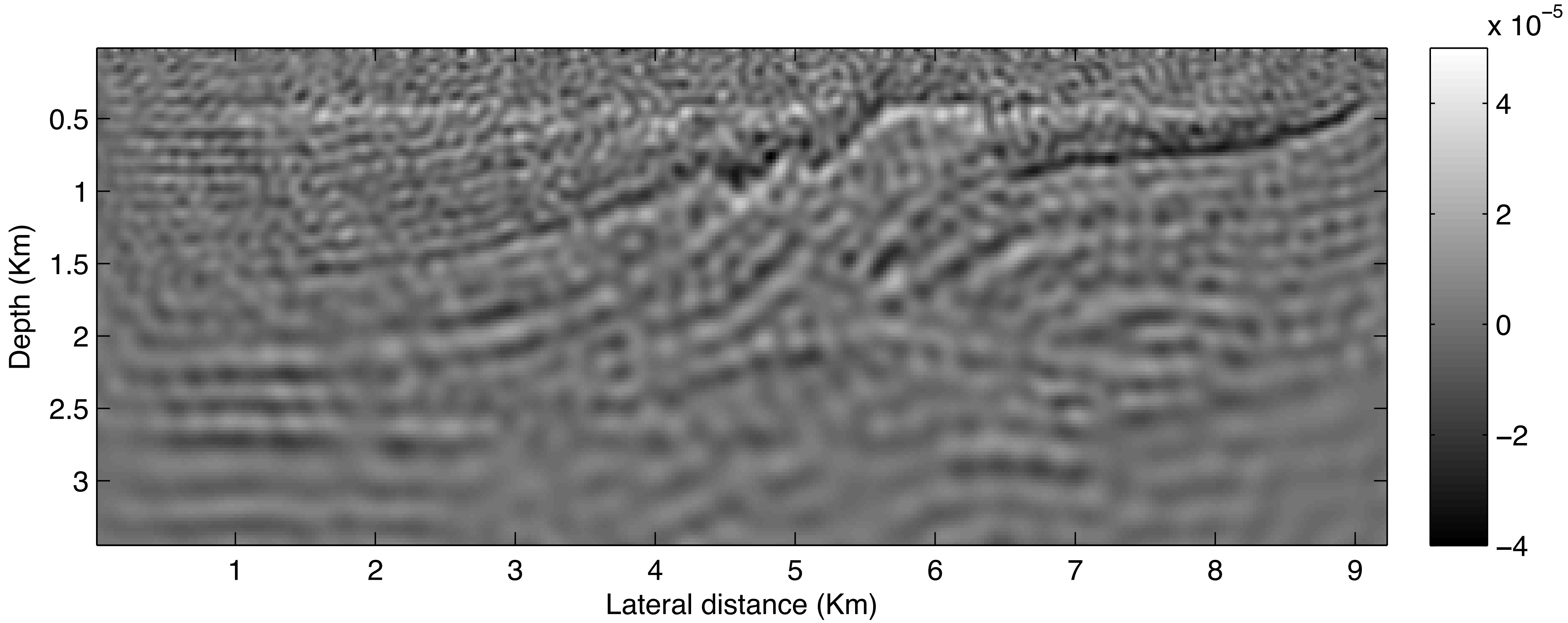
192 shots and 10 frequencies




Size of sampling matrix: 737280 x 49152

# L2 for supershots

8 supershots and 3 frequencies are used




Size of sampling matrix: 9216X49152

[Wang & Sacchi, '07]

# Sparse recovery

Least-squares migration with *sparsity* promotion

$$\delta\widetilde{\mathbf{m}} = \mathbf{S}^* \arg\min_{\delta\mathbf{x}} \frac{1}{2}\|\delta\mathbf{x}\|_{\ell_1} \quad \text{subject to} \quad \|\delta\underline{\mathbf{d}} - \nabla\boldsymbol{\mathcal{F}}[\mathbf{m}_0;\underline{\mathbf{Q}}]\mathbf{S}^*\delta\mathbf{x}\|_2 \leq \sigma$$

$$\delta\mathbf{x} \quad = \quad \text{Sparse curvelet-coefficient vector}$$

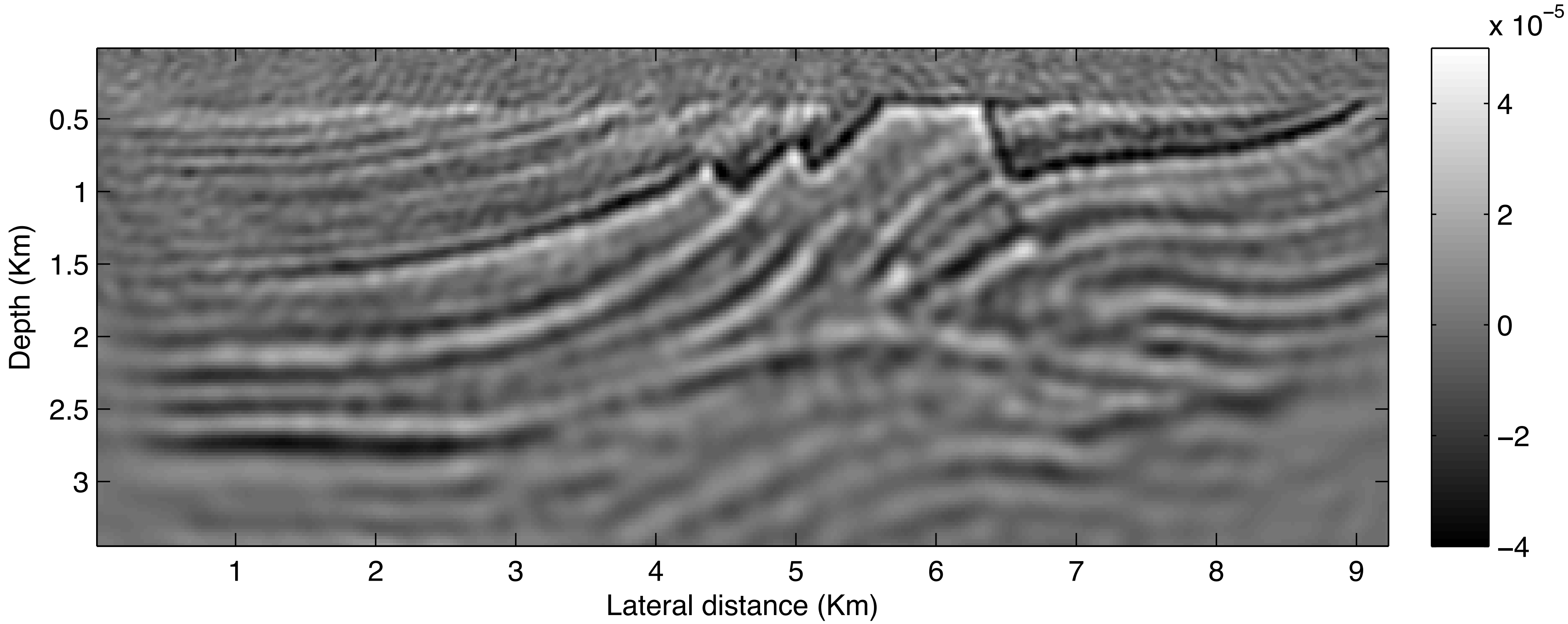
$$\mathbf{S}^* \quad = \quad \text{Curvelet synthesis}$$

leads to *significant* speedup as long as

$$n^{\ell_1}_{PDE} \times K \ll n^{\ell_2}_{PDE} \times n_f \times n_s$$

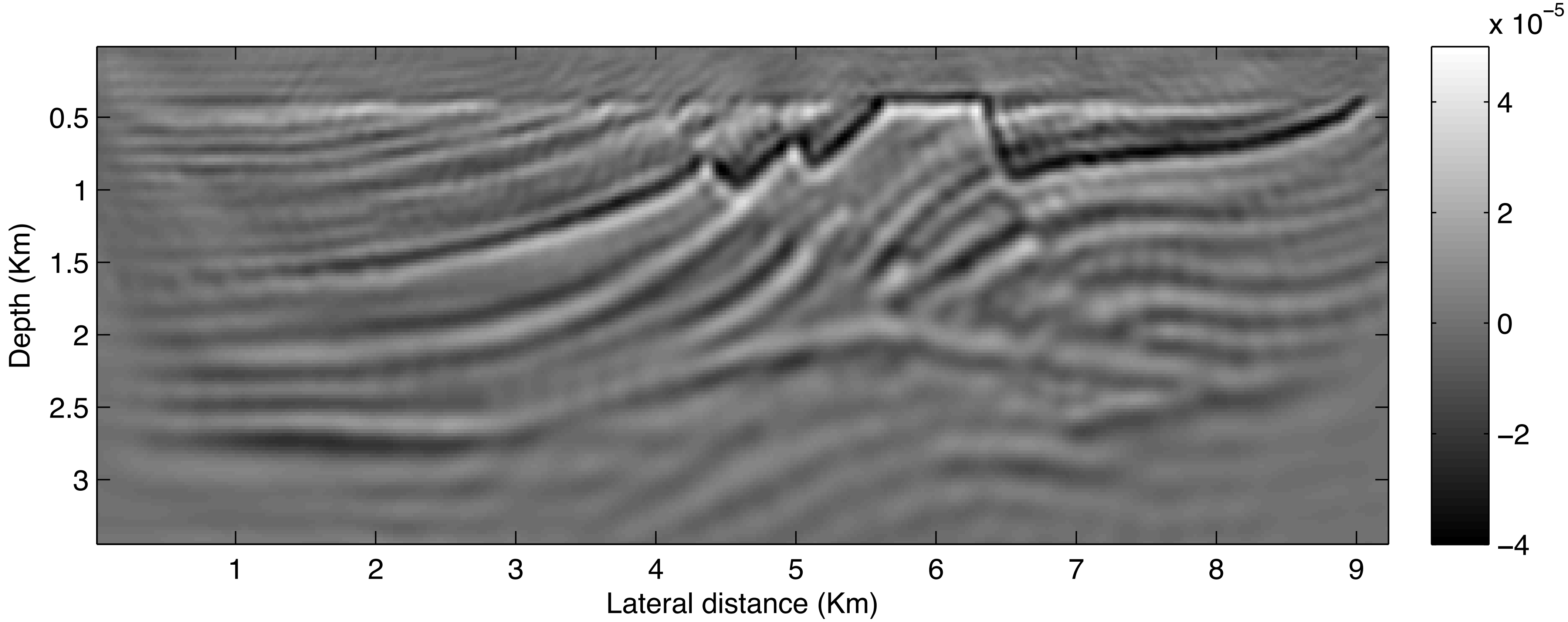
# L1 with small batch

8 supershots and 3 frequencies are used



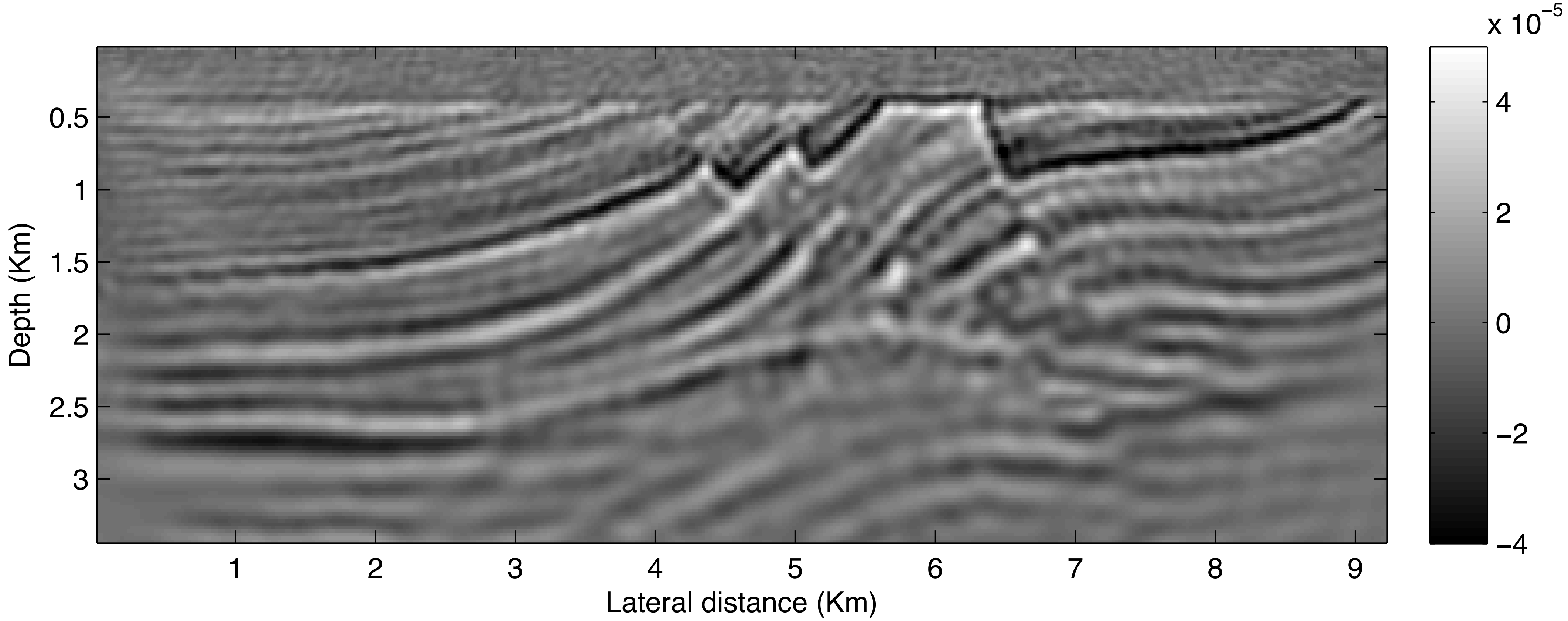
# L2 for all shots

192 shots and 10 frequencies



# Different batch size

15 supershots and 7 frequencies



# Observations

- Reconstruct images from smaller *under-determined* system with *sparse* recovery
- Sparse optimization is possible because of small batch size

# Compressive updates

**Algorithm 1**: Gauss Newton with sparse updates

**Result**: Output estimate for the model $\mathbf{m}$

$\mathbf{m} \longleftarrow \mathbf{m}_0$; $k \longleftarrow 0$ ; `// initial model`

**while** not converged **do**

$\quad \delta\mathbf{m}^k \longleftarrow \mathbf{S}^* \arg\min_{\mathbf{x}} \frac{1}{2}\|\delta\underline{\mathbf{d}}^k - \nabla\boldsymbol{\mathcal{F}}[\mathbf{m}^k;\underline{\mathbf{Q}}^k]\mathbf{S}^*\mathbf{x}\|_2^2 \quad \text{s.t.} \quad \|\mathbf{x}\|_1 \leq \tau^k$

$\quad \mathbf{m}^{k+1} \longleftarrow \mathbf{m}^k + \delta\mathbf{m}^k$ ; `// update with linesearch`

$\quad k \longleftarrow k + 1$;

**end**

[van den Berg & Friedlander, '08]

# Renewals

Use *different* batch of sources and frequencies for each *subproblem*

Requires *fewer* PDE solves for each GN *subproblem*...

- motivated by *stochastic approximation* [Nemirovski, '09]
- *justifies* approach by Krebs *et.al.*, 2009

# Combined approach

Leverage findings from *sparse recovery* & *compressive sensing*

- consider *phase-encoded* Gauss-Newton updates as separate "*compressive-sensing* /$\ell_1$regularized experiments"
- remove *interferences* by *curvelet-domain sparsity* promotion
- remove *bias* by selecting *new supershots* for each GN *iteration*

# Example

Marmousi model:

- 128x384 with a mesh size of 24 meters
- 192 shots and 384 receivers with offset = 3 X depth
- 3.6s recording time for Marmousi

*Explicit* Time-harmonic Helmholtz solver

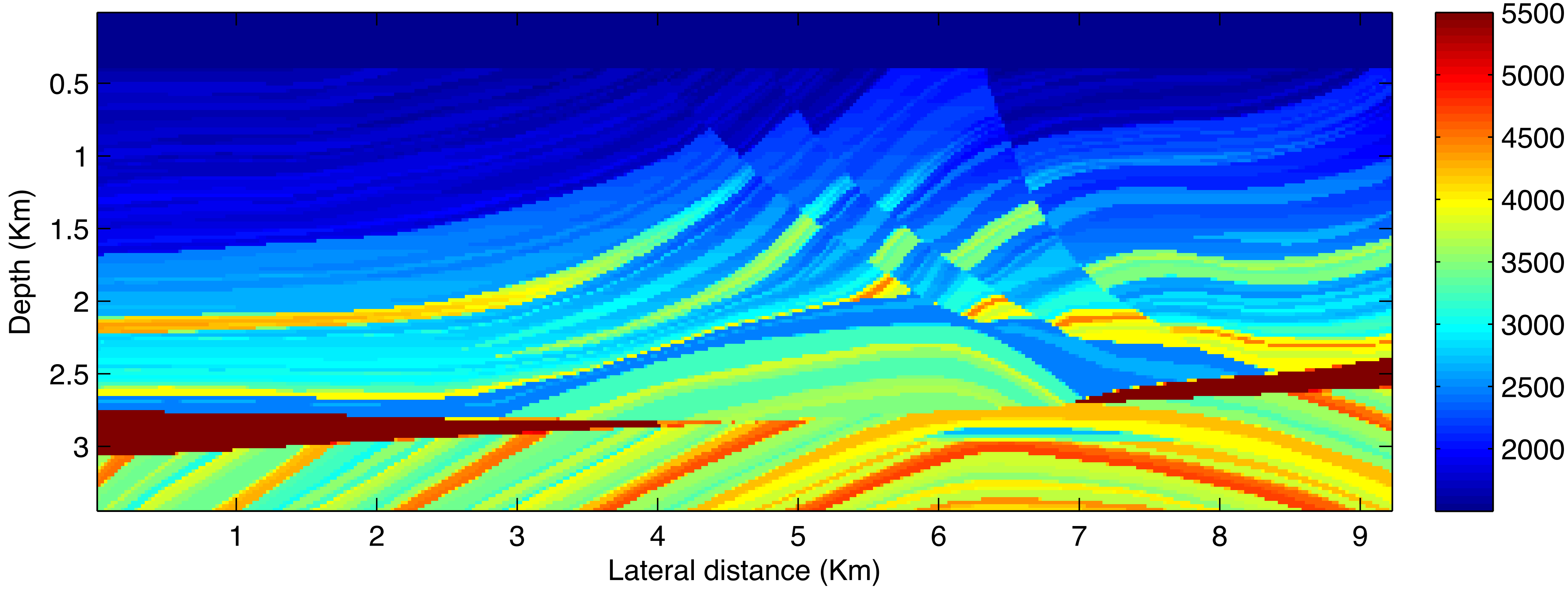
- 9-point finite difference
- absorbing boundary condition
- 12 Hz Ricker source wavelet

# Example

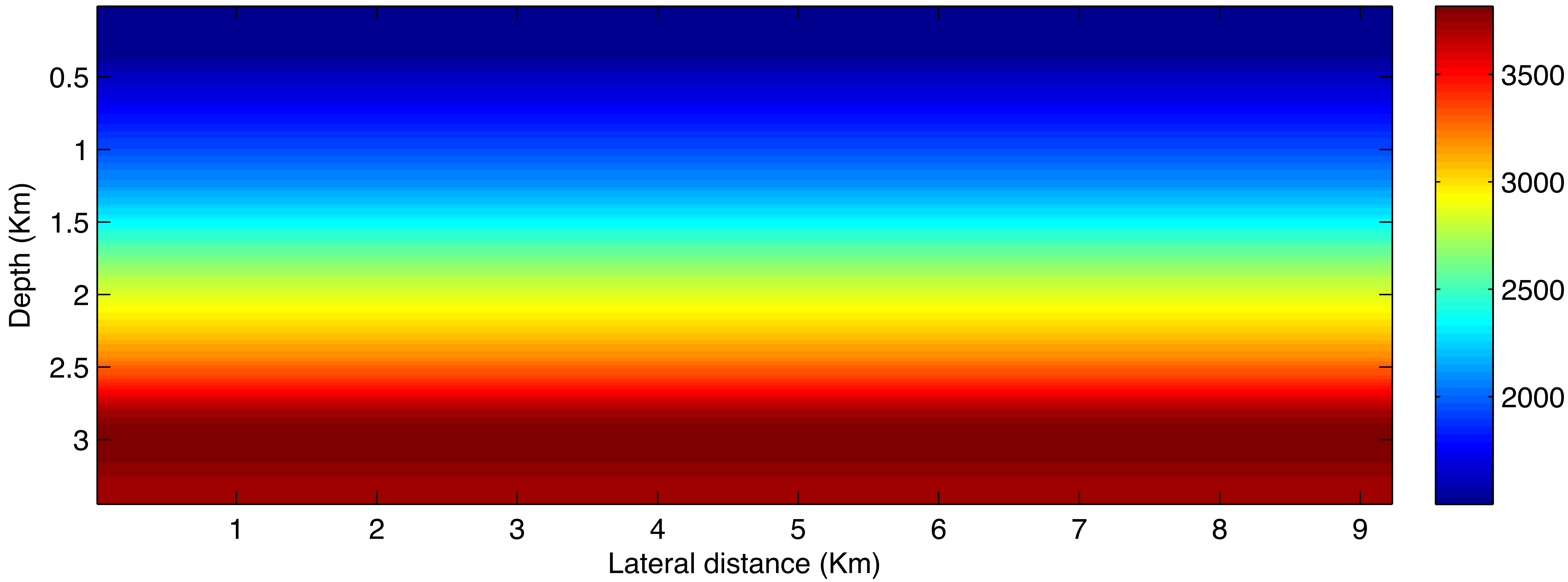
FWI specs:

- Frequency continuation over 10 bands
- 10 *frequencies* for each frequency band
- Start from 2.2 Hz

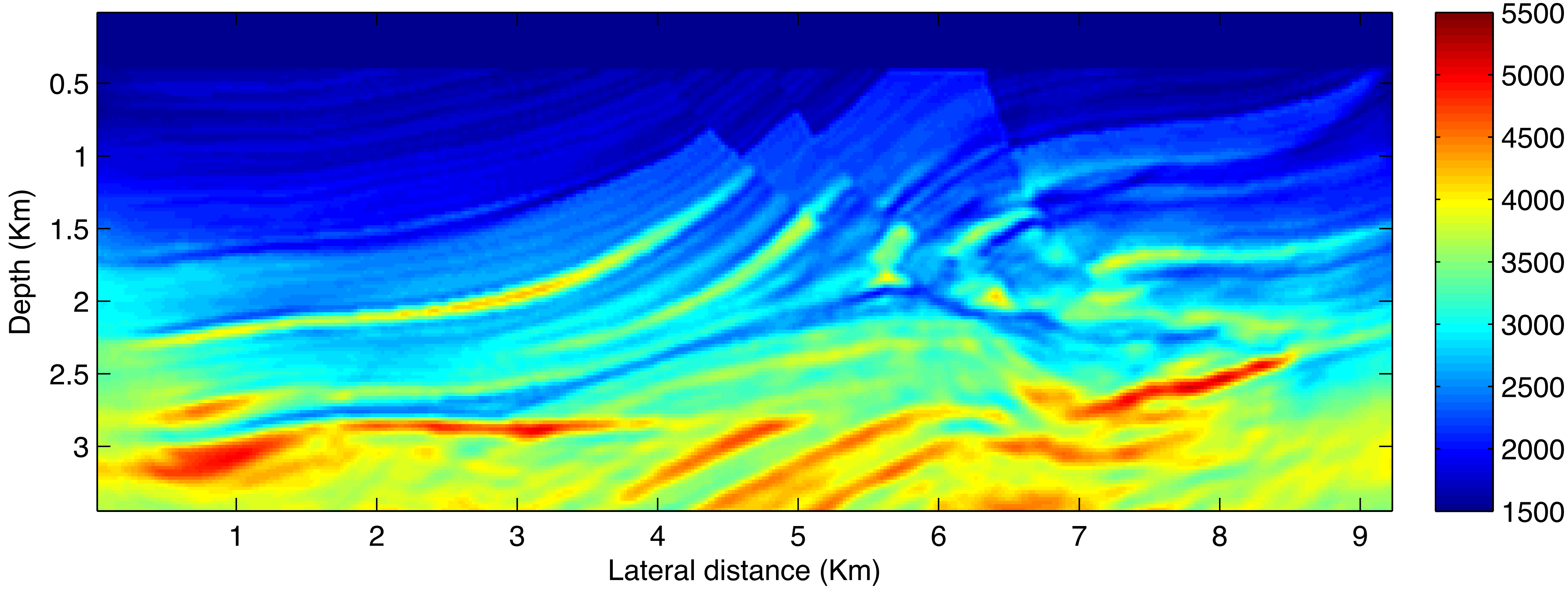
# True model


0.5
1
1.5
2
2.5
3
Depth (Km)
1
2
3
4
5
6
7
8
9
Lateral distance (Km)
5500
5000
4500
4000
3500
3000
2500
2000

# Initial model

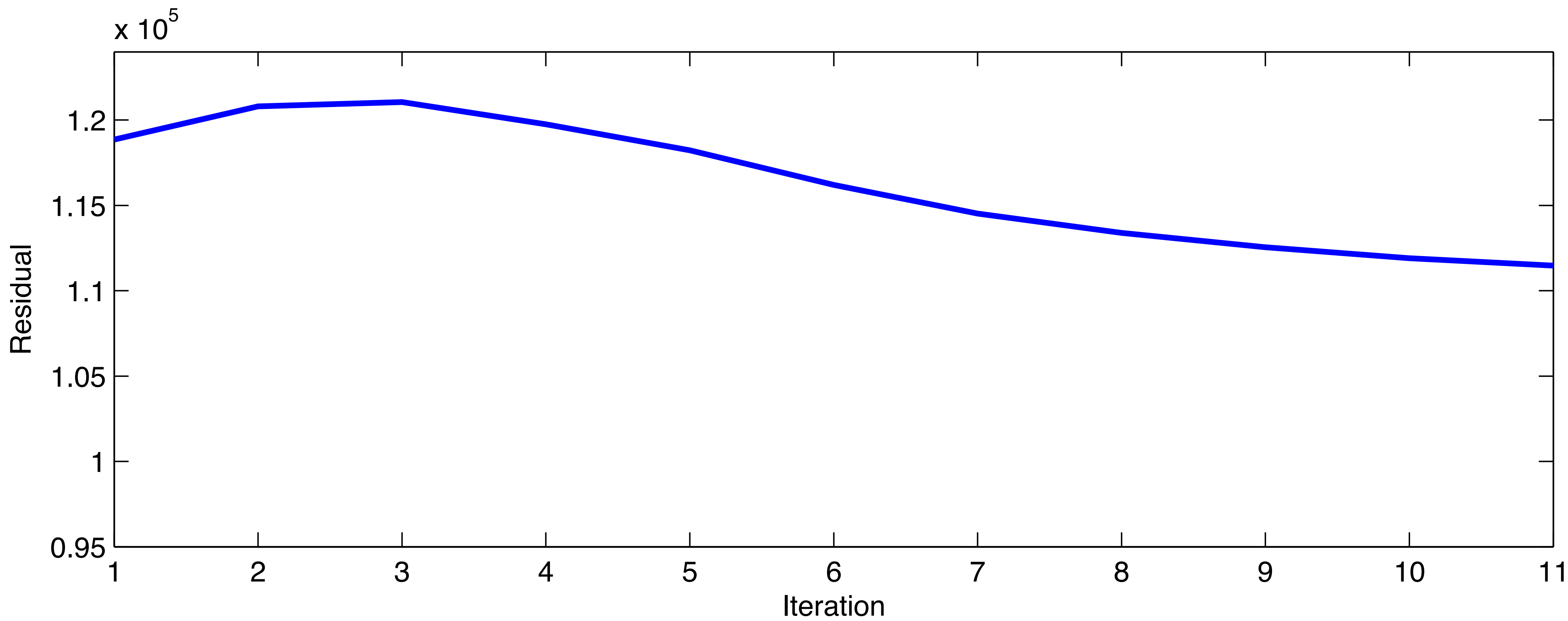


# L2 with *all* shots

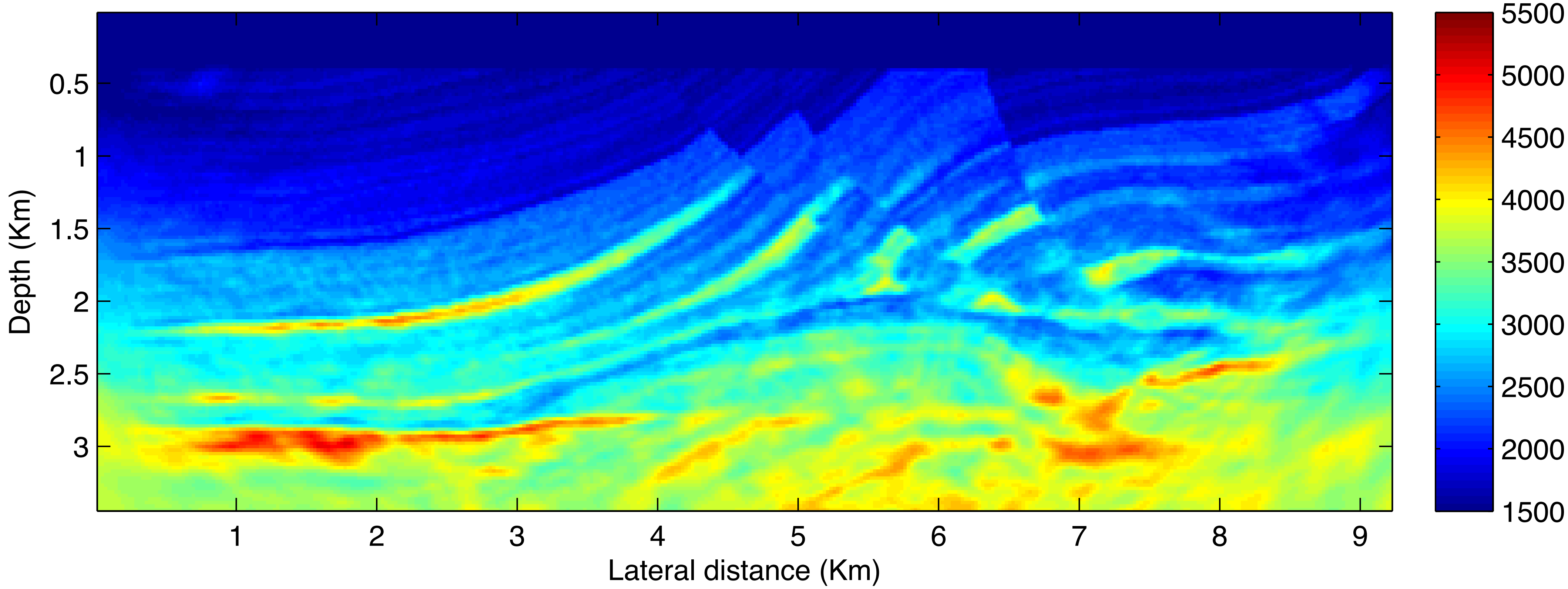



10f X192s X 45 X 2 = 172800 PDEs

# *Model* residual



# L1 without renewals




Speed up: X10

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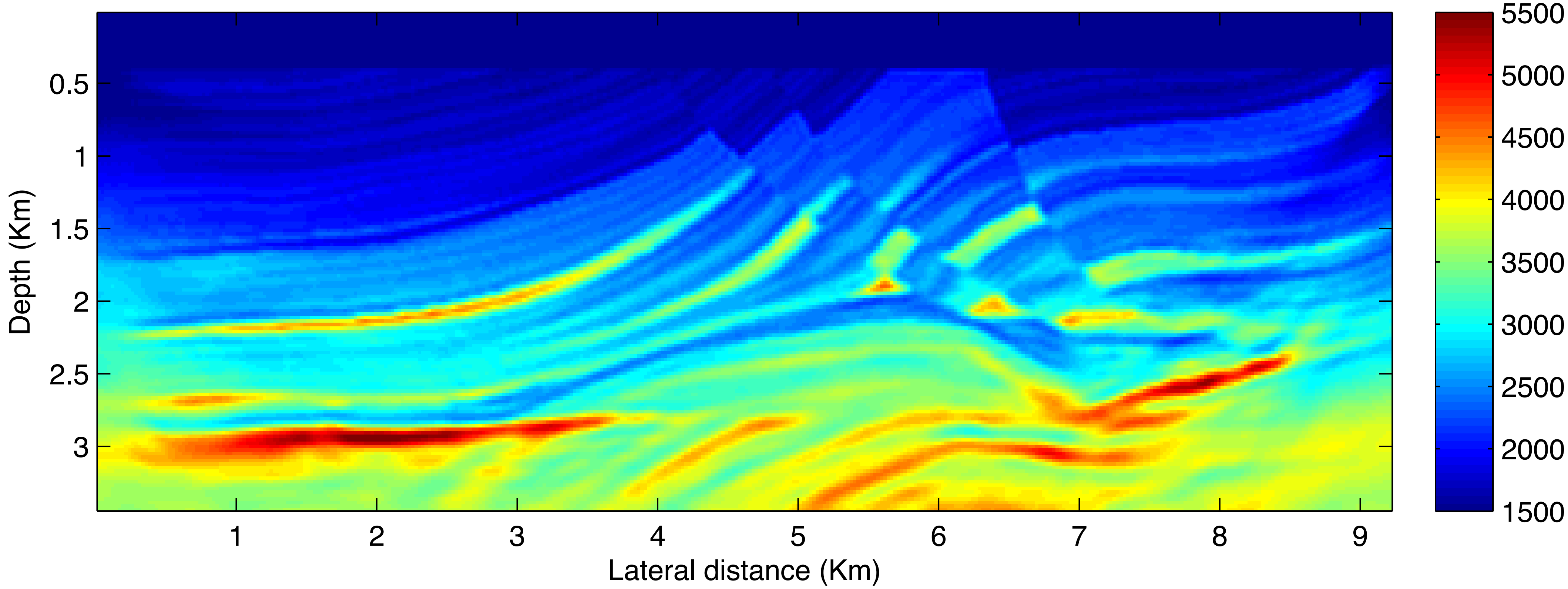
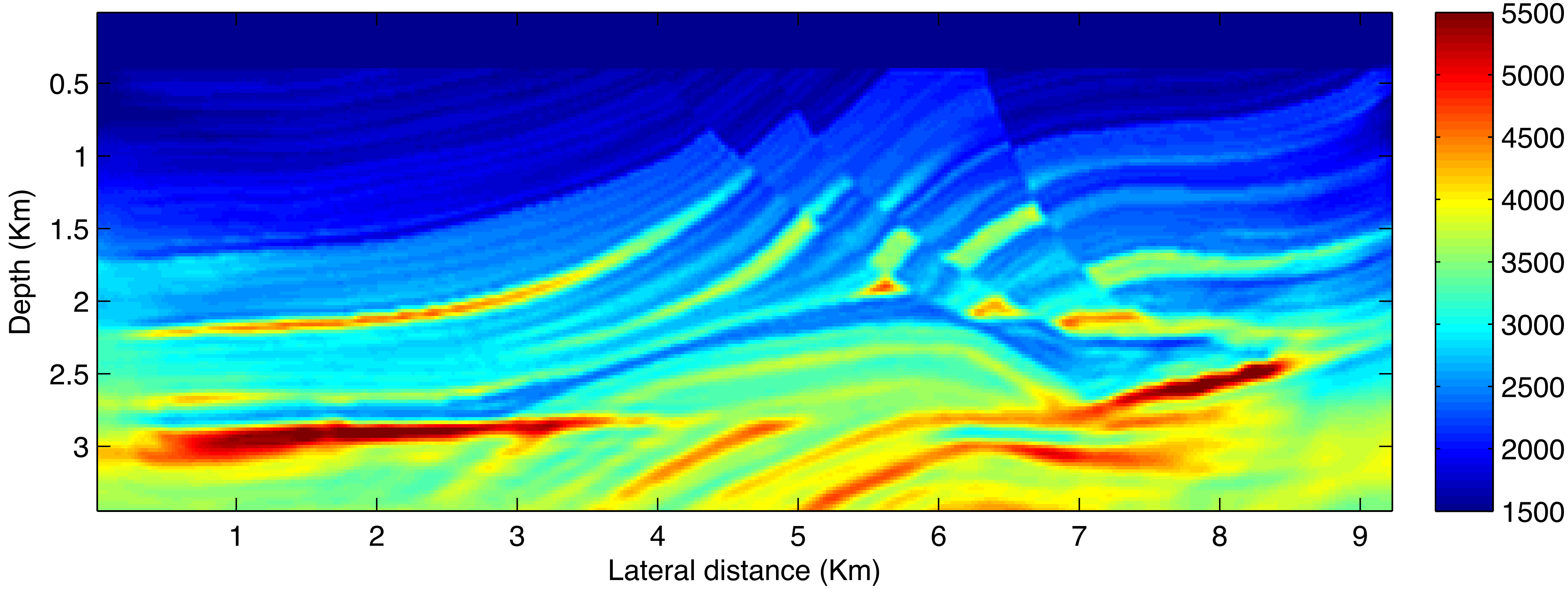
# L1 with renewals



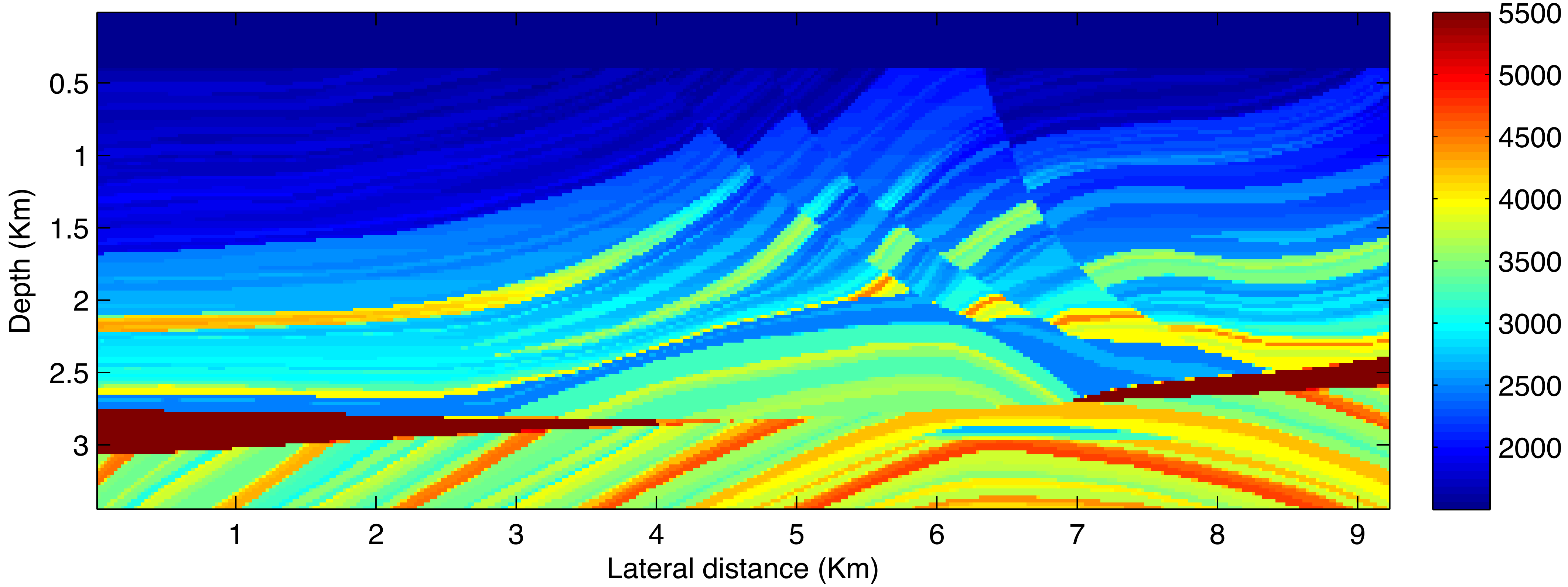

Speed up: X10

# L1 with renewals



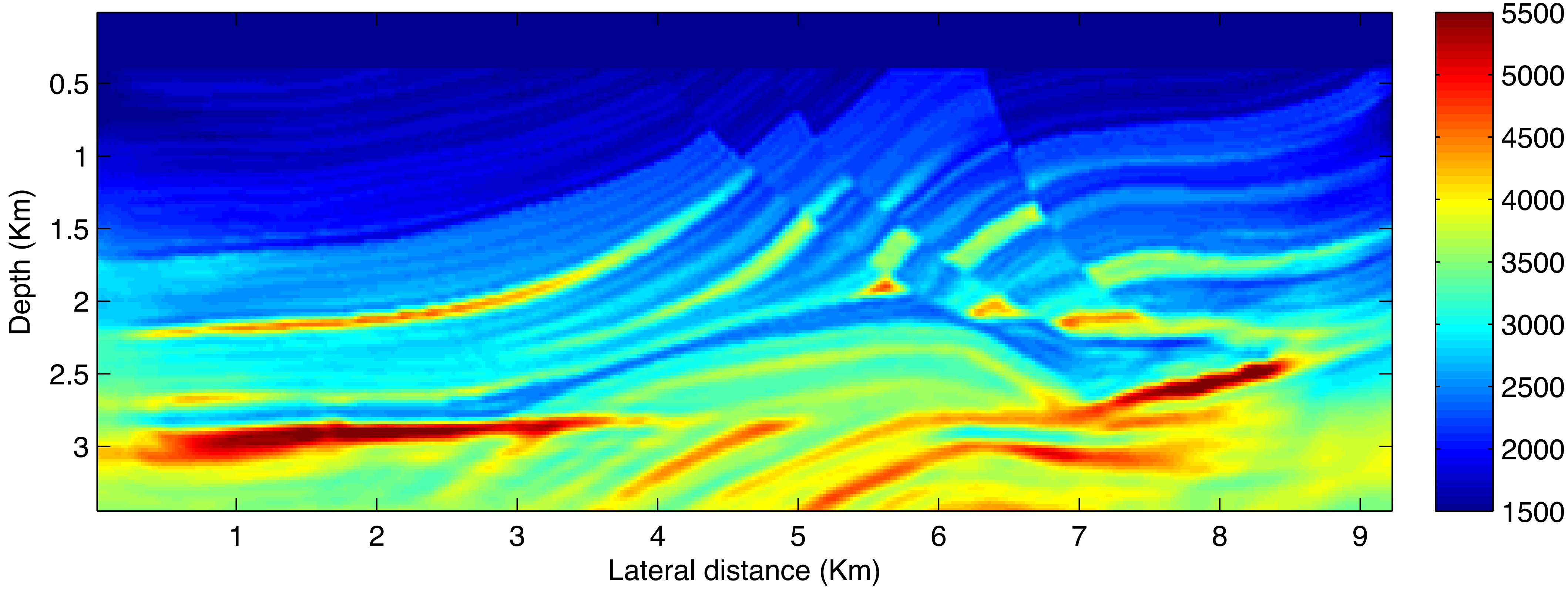

Speed up: X 3

# True result



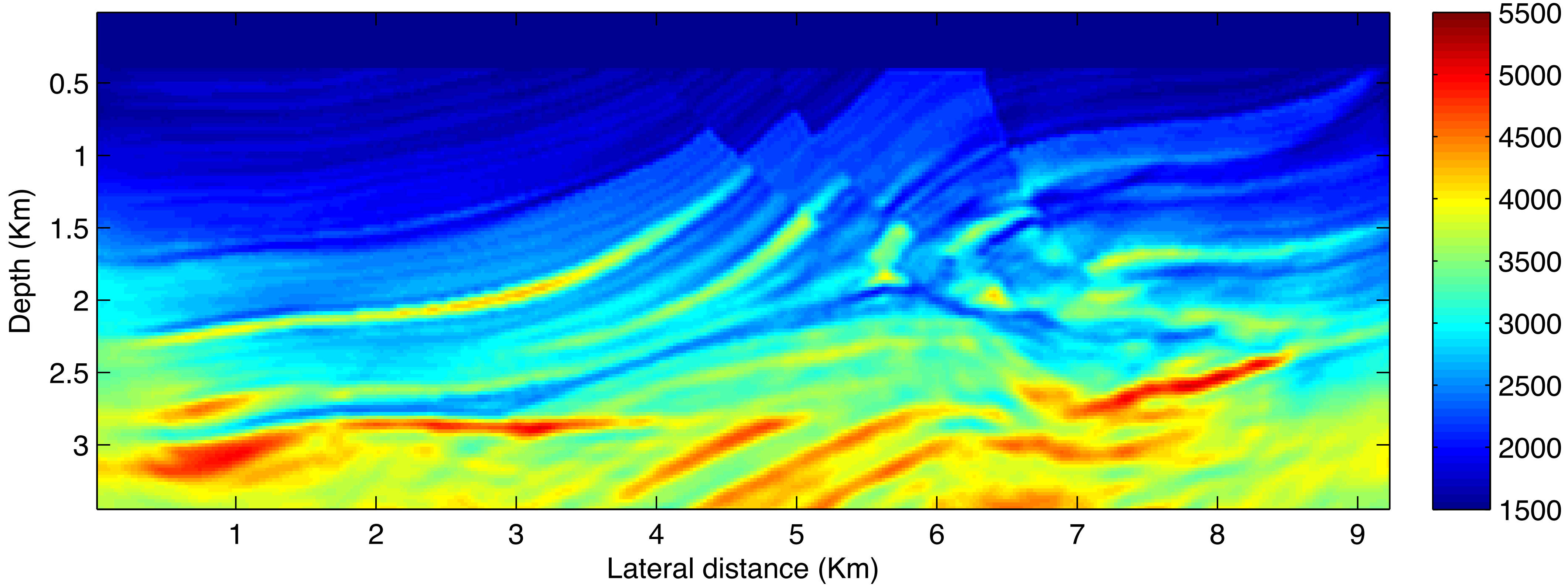
Depth (Km)
0.5
1
1.5
2
2.5
3
1
2
3
4
5
6
7
8
9
Lateral distance (Km)
5500
5000
4500
4000
3500
3000
2500
2000

# L1 with small batch

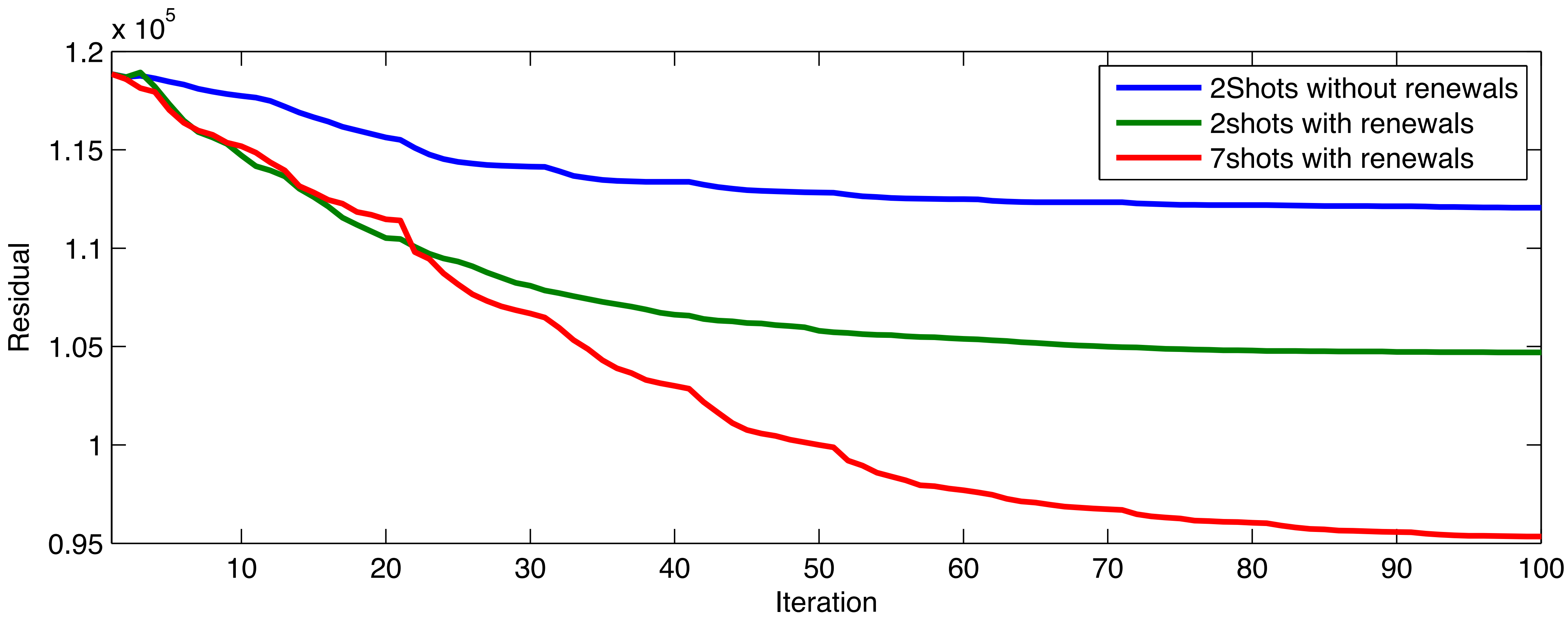



Speed up: X 3

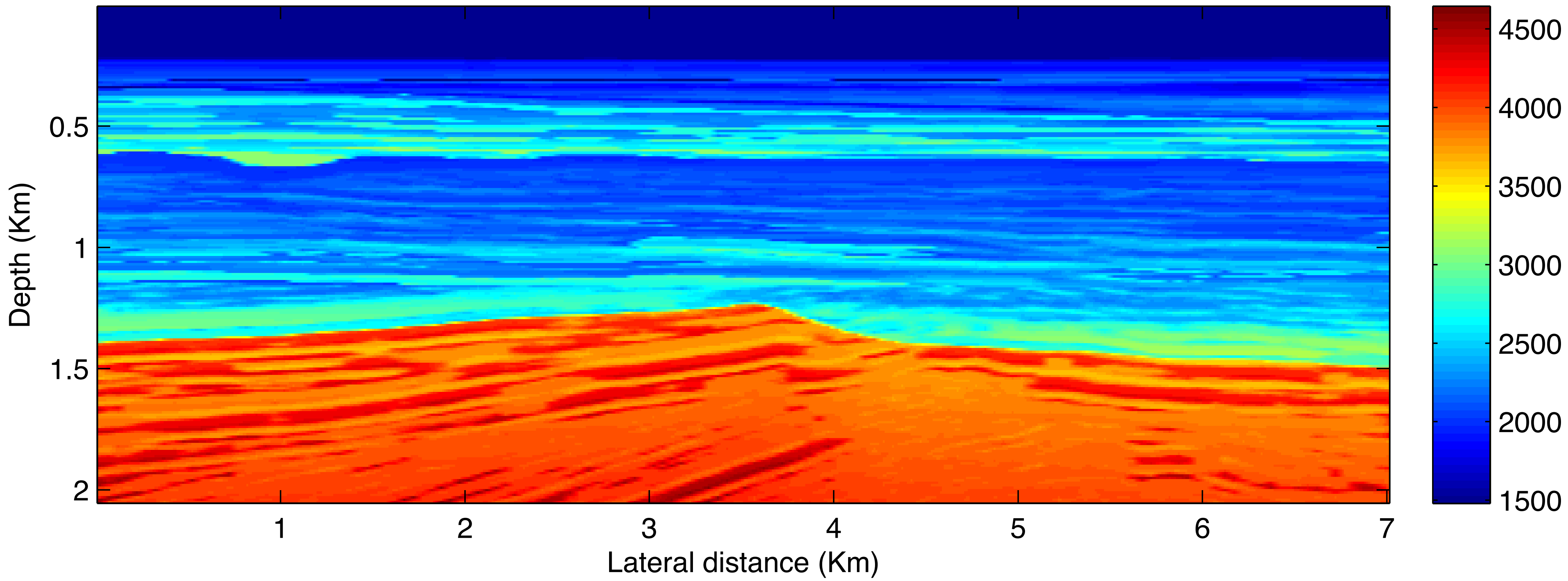
# L2 with *all* shots



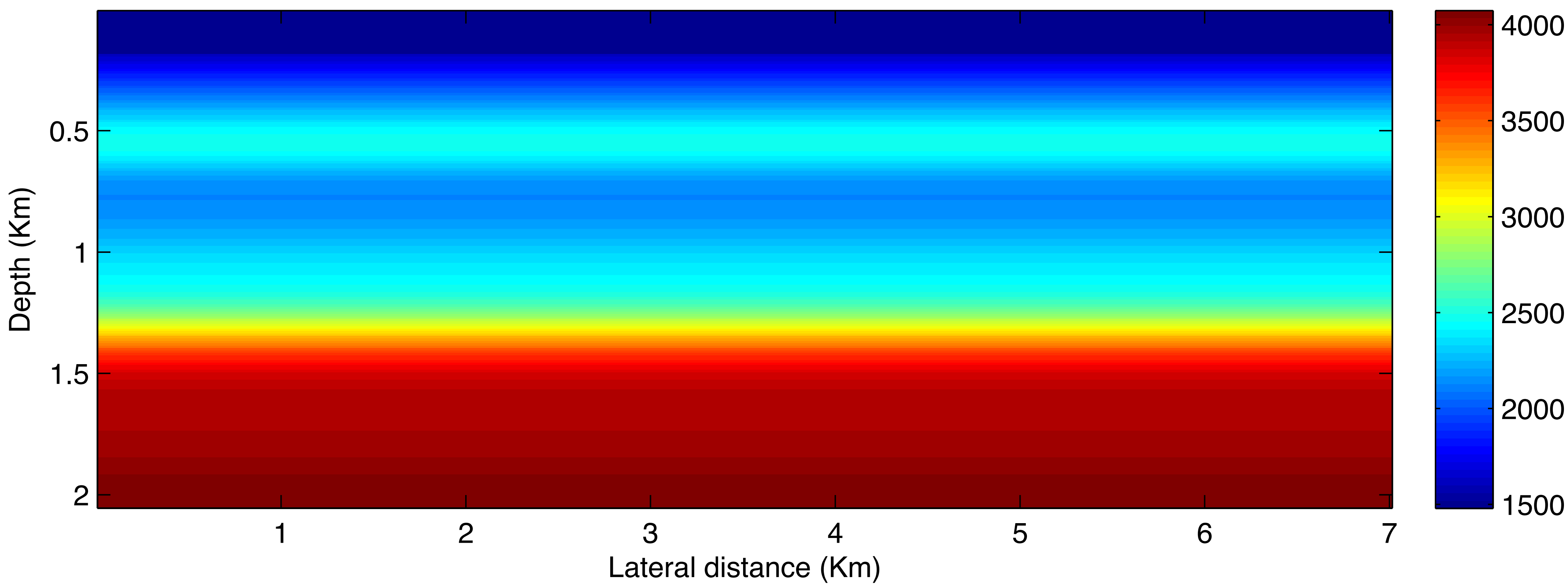
# Model residual



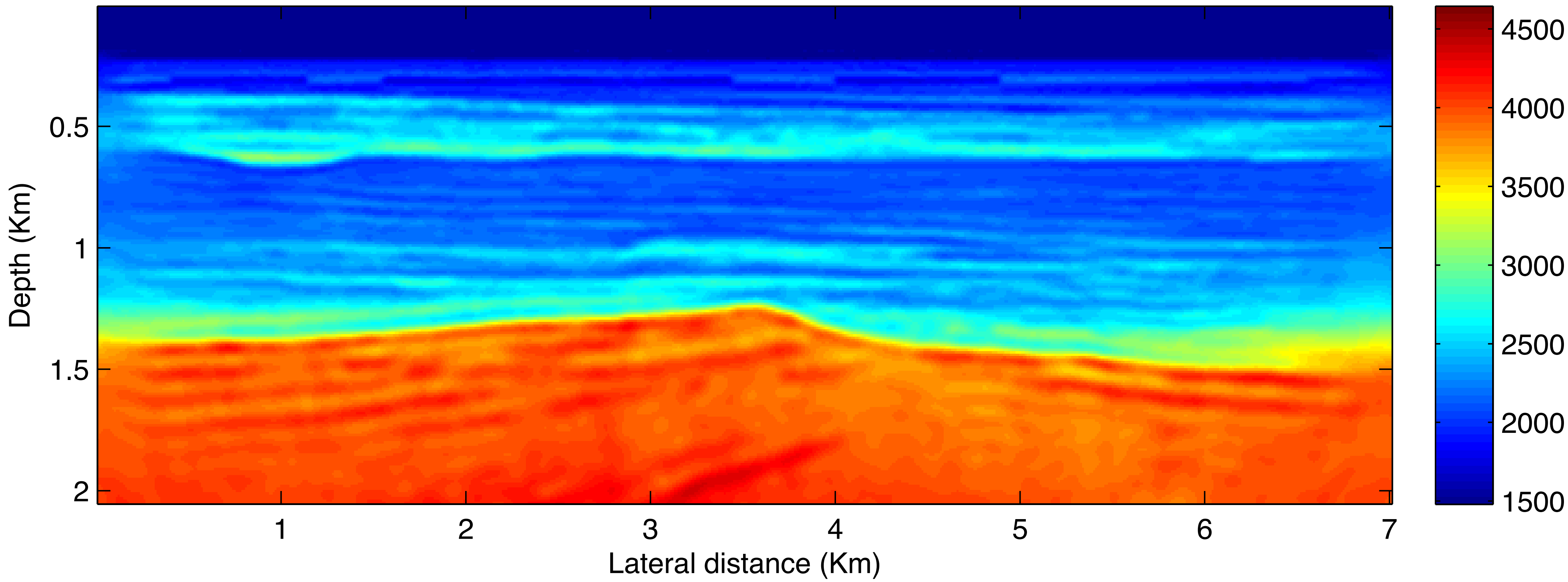
# True model


0.5
1
1.5
2
Depth (Km)
1
2
3
4
5
6
7
Lateral distance (Km)
4500
4000
3500
3000
2500
2000
1500

# Initial model

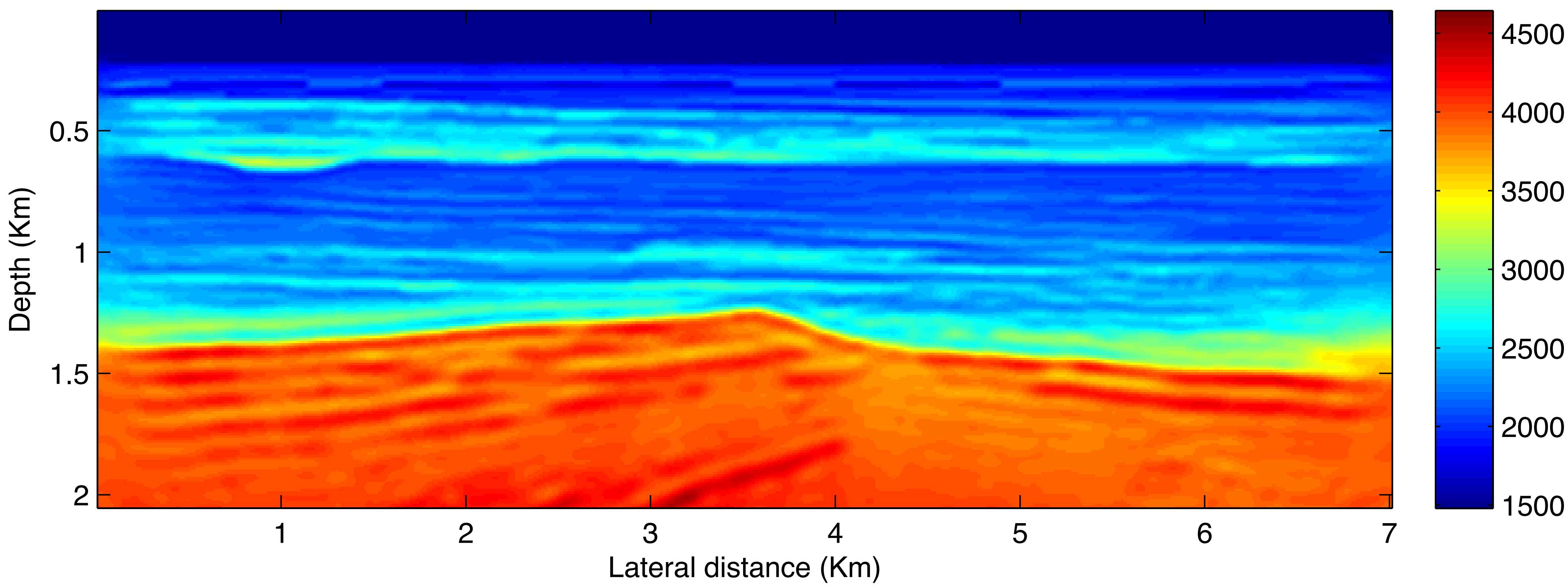


# Inverted result with 1 super shot



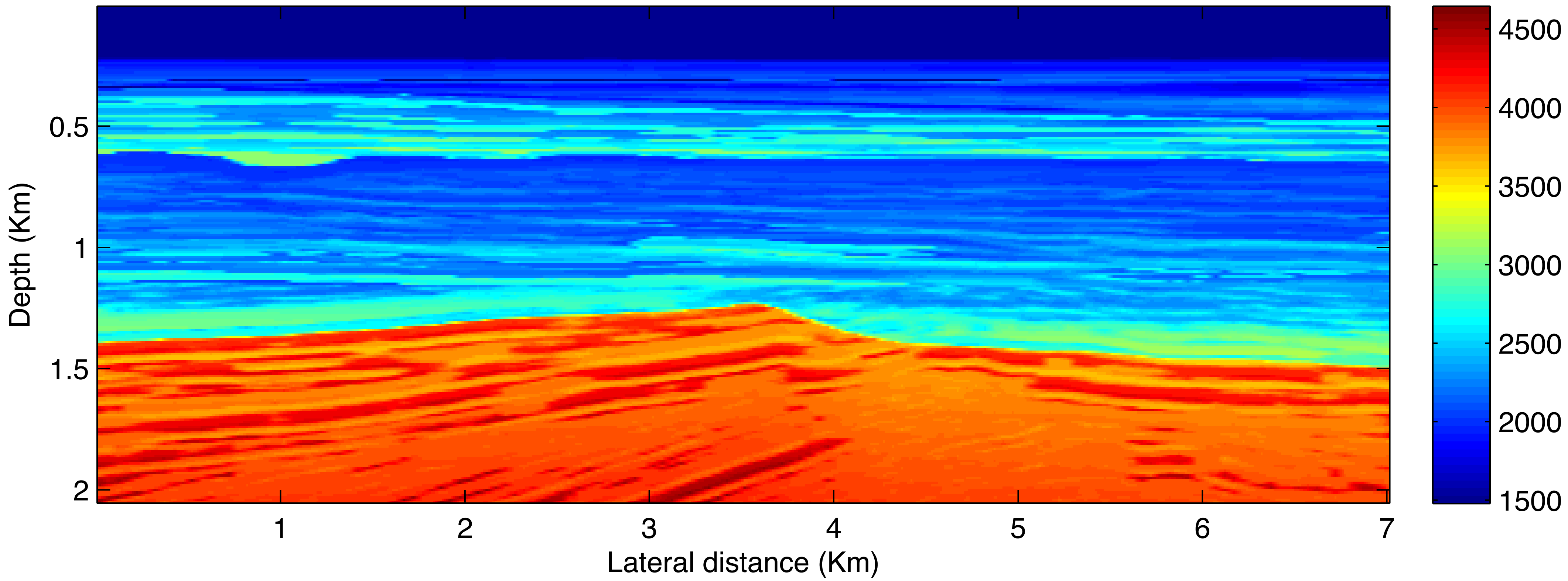

Start from 1.7 Hz

# Inverted result with 7 super shot




Start from 1.7 Hz

# True model


0.5
1
1.5
2
Depth (Km)
1
2
3
4
5
6
7
Lateral distance (Km)
4500
4000
3500
3000
2500
2000
1500

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# Observations

- Reduce the cost of FWI by using super shots
- Recover high resolution Gauss-Newton updates by imposing Curvelet domain sparsity

# Conclusions

- *sparse* recovery offsets *random interferences* due to *source encoding*
- *high-quality & high-resolution inversions* have been *achieved with significant accelerations*
- *no* need for additional *migration step*
- *improvements* come from *sparsity promotion & curvelets*
- indications that the *curse of dimensionality* can be removed...

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- Authors of CurveLab.

SINBAD

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# Thank you

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# Further reading

**Compressive sensing**

- *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information by Candes, 06*.
- *Compressed Sensing* by D. Donoho, '06
- Curvelets and Wave Atoms for Mirror-Extended Images by L. Demanet, L. Ying, 07.

**Simultaneous acquisition**

- *A new look at simultaneous sources* by Beasley et. al., '98.
- *Changing the mindset in seismic data acquisition* by Berkhout '08.

**Simultaneous simulations, imaging, and full-wave inversion:**

- *Faster shot-record depth migrations using phase encoding* by Morton & Ober, '98.
- *Phase encoding of shot records in prestack migration* by Romero et. al., '00.
- *Efficient Seismic Forward Modeling using Simultaneous Random Sources and Sparsity* by N. Neelamani et. al., '08.
- *Compressive simultaneous full-waveform simulation* by FJH et. al., '09.
- *Fast full-wavefield seismic inversion using encoded sources* by Krebs et. al., '09
- *Randomized dimensionality reduction for full-waveform inversion* by FJH & X. Li, '10

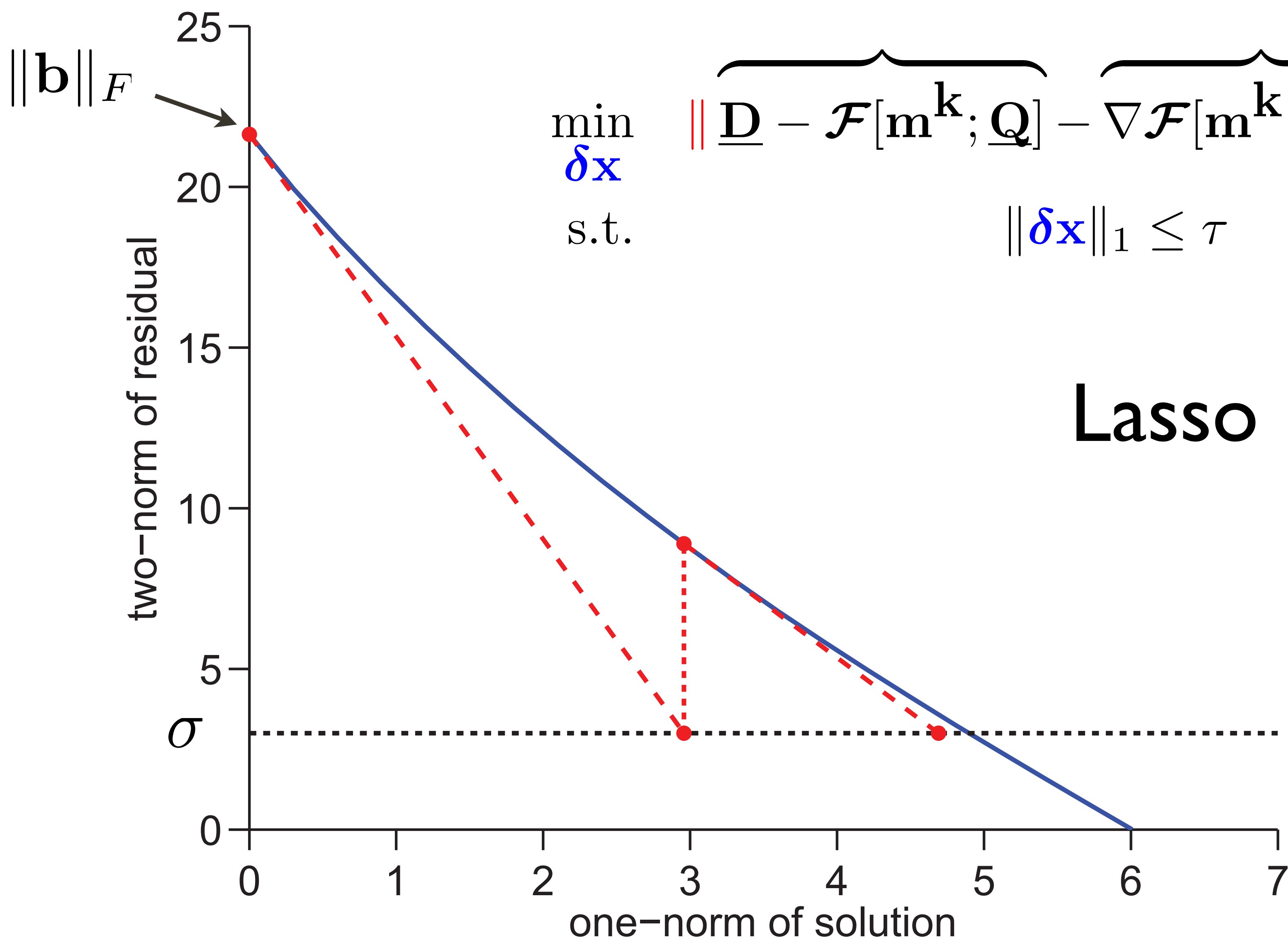
**Stochastic optimization and machine learning:**

- *A Stochastic Approximation Method* by Robbins and Monro, 1951
- *Neuro-Dynamic Programming* by Bertsekas, '96
- *Robust stochastic approximation approach to stochastic programming* by Nemirovski et. al., '09
- *Stochastic Approximation and Recursive Algorithms and Applications* by Kushner and Lin
- *Stochastic Approximation approach to Stochastic Programming* by Nemirovski
- *An effective method for parameter estimation with PDE constraints with multiple right hand sides. by* Eldad Haber, Matthias Chung, and Felix J. Herrmann. '10

# Picking Lasso Parameter

$$\min_{\boldsymbol{\delta}\mathbf{x}} \; \| \overbrace{\underline{\mathbf{D}} - \mathcal{F}[\mathbf{m}^{\mathbf{k}}; \underline{\mathbf{Q}}]}^{\mathbf{b}} - \overbrace{\nabla\mathcal{F}[\mathbf{m}^{\mathbf{k}}; \underline{\mathbf{Q}}]\mathbf{S}^*}^{\mathbf{A}} \, \boldsymbol{\delta}\mathbf{x} \|_F^2$$

$$\text{s.t.} \quad \|\boldsymbol{\delta}\mathbf{x}\|_1 \leq \tau$$




Lasso