Introduction

From the measured seismic data, the location and the amplitude of reflectors can be determined via a migration algorithm. Classically, following Claerbout's imaging principle [2], a reflector is located at the position where the source's forward-propagated wavefield correlates with the backward-propagated wavefield of the receiver data. Lailly and Tarantola later showed that this imaging principle is an instance of inverse problems, with the associated migration operator formulated via a least-squares functional; see [6, 12, 13]. Furthermore, they showed that the migrated image is associated with the gradient of this functional with respect to the image. If the solution of the least-squares functional is done iteratively, the correlation-based image coincides up to a constant with the first iteration of a gradient method. In practice, this migration is done either in the time domain or in the frequency domain.

In the frequency-domain migration, the main bottleneck thus far, which renders its full implementation to large scale problems, is the lack of efficient solvers for computing wavefields. Robust direct methods easily run into excessive memory requirements as the size of the problem increases. On the other hand, iterative methods, which are less demanding in terms of memory, suffered from lack of convergence. During the past years, however, progress has been made in the development of an efficient iterative method [4, 3] for the frequency-domain wavefield computations. In this paper, we will show the significance of this method (called MKMG) in the context of the frequency-domain migration, where multi-shot-frequency wavefields (of order of 10,000 related wavefields) need to be computed.

Theory

In the frequency domain, seismic waveform inversion is typically performed by updating the velocity estimate by the update [9]

$$\delta\mathbf{m} = \Re\left(\sum_{i_\omega=1}^{n_\omega}\sum_{i_s=1}^{n_s}\mathbf{u}_{i_s,i_\omega}[\mathbf{m}_0]\odot\mathbf{v}_{i_s,i_\omega}[\mathbf{m}_0]\right) = \Re\left(\mathrm{diag}(\mathbf{U}\mathbf{V}^*)\right), \tag{1}$$

during an iterative gradient method. In (1), the symbol "$\odot$" denotes the Hadamard (entry-wise) product of two vectors. The vectors $\mathbf{u}_{i_s,i_\omega}$ are the solution of the forward modeling based on the smooth background $\mathbf{m}_0$,

$$H[\mathbf{m}_0,\omega_{i_\omega}]\mathbf{u}_{i_s,i_\omega} = \mathbf{b}_{i_s}, \tag{2}$$

where H is the discretized Helmholtz operator for an angular frequency ω and $\mathbf{b}$ the right-hand side (source) vector. For multi-shot-frequency simulations, the vectors $\mathbf{u}_{i_s,i_\omega}$ are compactly organized in the columns (the shot direction) and in the rows (the frequency direction) of the matrix $\mathbf{U}$. The vectors $\mathbf{v}_{i_s,i_\omega}$ are the back-propagated wavefields, which are the solution of the adjoint system

$$H^*[\mathbf{m}_0,\omega_{i_\omega}]\mathbf{v}_{i_s,i_\omega} = \delta\mathbf{d}_{i_s,i_\omega}, \tag{3}$$

where H^* is the adjoint of H and $\delta\mathbf{d}_{i_s,i_\omega} = \mathbf{d}_{i_s,i_\omega} - \mathbf{D}\mathbf{u}_{i_s,i_\omega}$, with $\mathbf{D}$ a matrix, which restricts the wavefield $\mathbf{u}$ to the receiver positions. These vectors can also be organized in the same way as $\mathbf{u}$, which results in the matrix $\mathbf{V}$. Upto a constant, the first update of a gradient method is equivalent to the correlation-based migration [9]:

$$\delta\widetilde{\mathbf{m}} = \mathbf{K}^T\delta\mathbf{d}, \tag{4}$$

with $\mathbf{K}^T$ the migration operator, and $\delta\mathbf{d}$ the multi-shot-frequency data.

MKMG: Iterative method for the Helmholtz equation

Iterative solutions of Equations (2) and (3) can be computed efficiently by the multilevel Krylov-multigrid method (MKMG), introduced in [3], which is an improvement to the earlier method discussed in [4] and [10]. In MKMG, a combined action of an effective preconditioner and a deflation method is used to obtain fast convergence. In this case, the preconditioner is based on the damped Helmholtz operator (or called shifted Laplacian):

$$M := \nabla^2 - (1-0.5\hat{i})\left(\frac{\omega}{c}\right)^2, \quad \hat{i} = \sqrt{-1}, \tag{5}$$

which is solved approximately by one multigrid iteration. The action of this preconditioner shifts the eigenvalues of the Helmholtz equation to the positive half plane, making the preconditioned system definite (the real parts of eigenvalues are positive), and cluster in a circle with the center $P = (\frac{1}{2}, 0)$ and the radius $r = 0.5$ (see Figure 1:middle). Deflation of the form

$$\mathcal{P}_N := I - Z^T E^{-1} Y^T H M^{-1} + Z^T E^{-1} Y^T, \quad E = Y^T H M^{-1} Z, \tag{6}$$

is then applied to the preconditioned system. The middle term in the above operator shifts the eigenvalues to zeros, while the last term shifts them back towards one. Thus, in overall, the action of $\mathcal{P}_N$ clusters further the already clustered eigenvalues around one; this eigenvalue clustering is illustrated in Figure 1:right. The final eigenvalue property of the system $HM^{-1}\mathcal{P}_N$ is very favorable for an iterative method. If applied to the preconditioned linear system

$$HM^{-1}\mathcal{P}_N \tilde{x} = b, \quad x = M^{-1}\mathcal{P}_N \tilde{x}, \tag{7}$$

an iterative method will converge fast. Figure 2:left shows the convergence of the MKMG method applied to Equations (2) and (3), with the hard Marmousi velocity model, which clearly indicates the frequency-independent convergence. In Figure 2, the memory requirement for MKMG is compared with traditional direct methods based on LU factorizations, which shows that this new iterative method is less demanding in terms of memory. As these nice properties are generalized to 3-D, it becomes viable to do large scale seismic imaging based on this method.

Example

For an illustrative example, we consider migrating an image from a smooth velocity model using multi-shot-frequency data. This smooth model is generated from the hard Marmousi model. To generate the migrated image, 188 shots are used. The receivers are located at the same depth position as the source. In total, 751 receivers are used. The Helmholtz matrix H is generated by a high-order finite difference scheme based on [5], which is applied on a 751×201 uniform mesh. To avoid spurious non-physical reflections at the boundaries, dampling layers are added to the physical domain, making the computational domain even larger. In the frequency-domain migration, it is often sufficient to use only a few number of frequencies to produce a useful image [7], [11]. In our case, we use frequencies from 0.5 Hz to 5.0 Hz. Figure 3: right shows the migrated image. The close relations between the hard model and the migrated image can be clearly seen in Figure 4, where traces at two horizontal positions are shown ($x = 16$ and 2800 meter).

Performance wise, for each shot and frequency, about 5 seconds of CPU time are needed to compute one wavefield. This CPU time already includes the time to construct the information needed in the MKMG method, which is only fractional. Most of the time is spent in I/O process, since the current solver is based on the out-of-the core implementation. Starting from generating the data, about 9 hours are spent to get the image on a single processor PC with 4 Gb of RAM.

Conclusion

With the MKMG method, we have shown in this paper the possibility of performing migration in an efficient way. We note that the MKMG method is inherently parallelazable, because all integrated components in it are fully parallelizable. Also, frequency-domain migration can be made parallel both in the shot direction and the frequency direction. While in terms of performance, the result shown here is already a significant improvement compared to the traditional LU-based factorizations to compute the forward- and the back-propagated wavefields, implementing both the MKMG method and the migration algorithm in a parallel environment will lead to a further perfomance improvement. The parallelism of a frequency-domain migration algorithm is also conducive to compressive sampling, which allows sub-sampling in shots and frequencies; hence, only a few number of shots and frequencies are required. This will provide a rigorous framework for selecting subsamples of frequencies and shots, from which the image is recovered via a sparsity-promoting algorithm.

References

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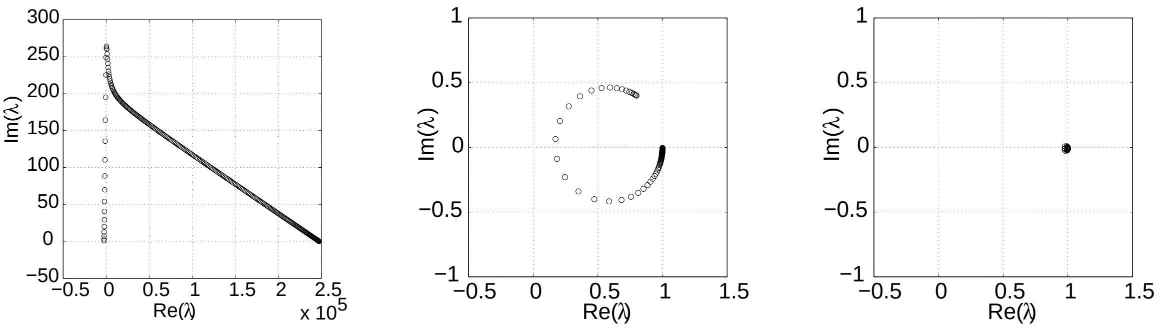

Figure 1: Eigenvalues of the Helmholtz equation before preconditioning (left), after preconditioning (center), and after applying deflation (right). With preconditioning and deflation combined, the eigenvalues are now clustered around one.

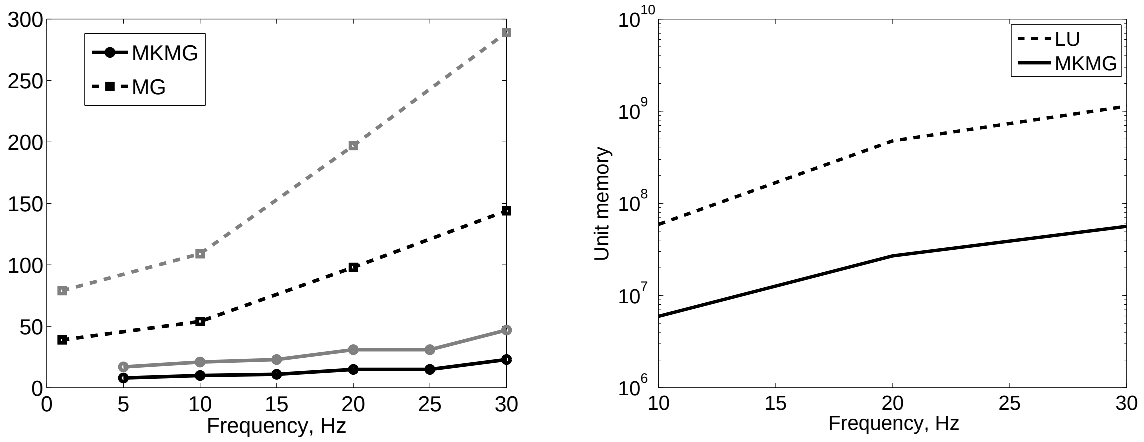

Figure 2: Convergence of the MKMG method for the 2D Helmholtz equation with the hard Marmousi velocity model (left figure). Compared to using only multigrid (MG) [4] and [10], the MKMG methods shows a significant improvement. MKMG also demands less memory requirement as compared to LU decomposition (right figure).

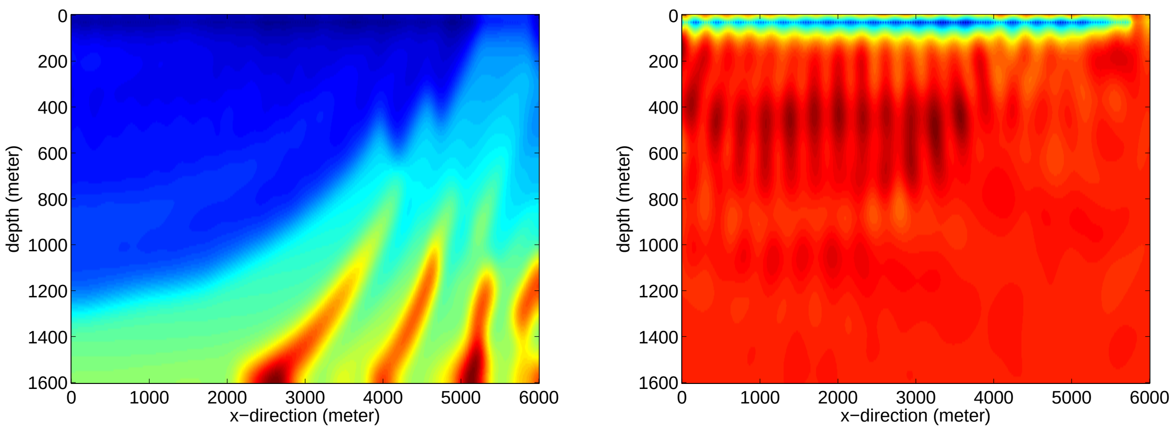

Figure 3: Upper part of Marmousi model: velocity after migration (left) and the updates (right)

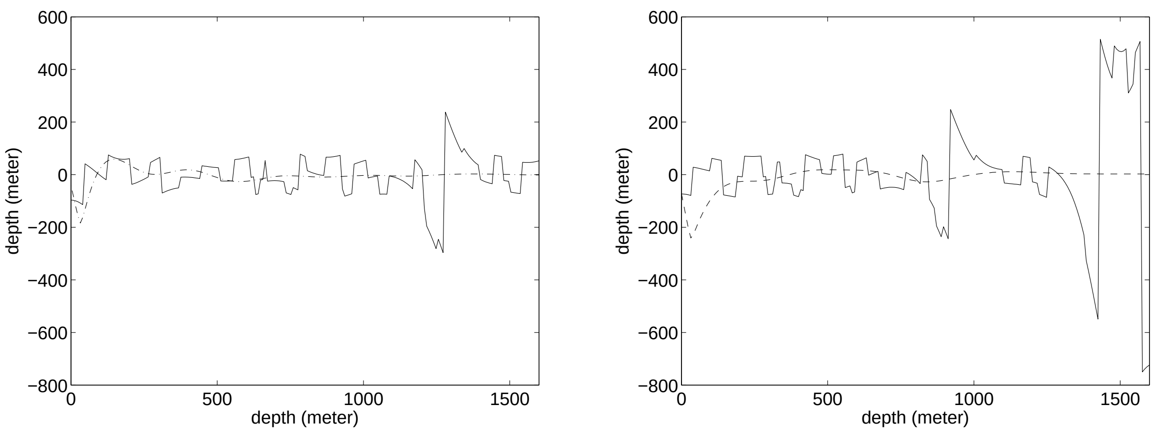

Figure 4: Traces of the true reflectivity (solid line) and the migrated image (dashed line) at $x = 16$ meter (left) and $x = 2800$ m (right).