Curvelet imaging & processing: sparseness constrained least-squares migration

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thanks to: Gilles, Peyman and Candes, Sacchi, Rickett, WaveLab

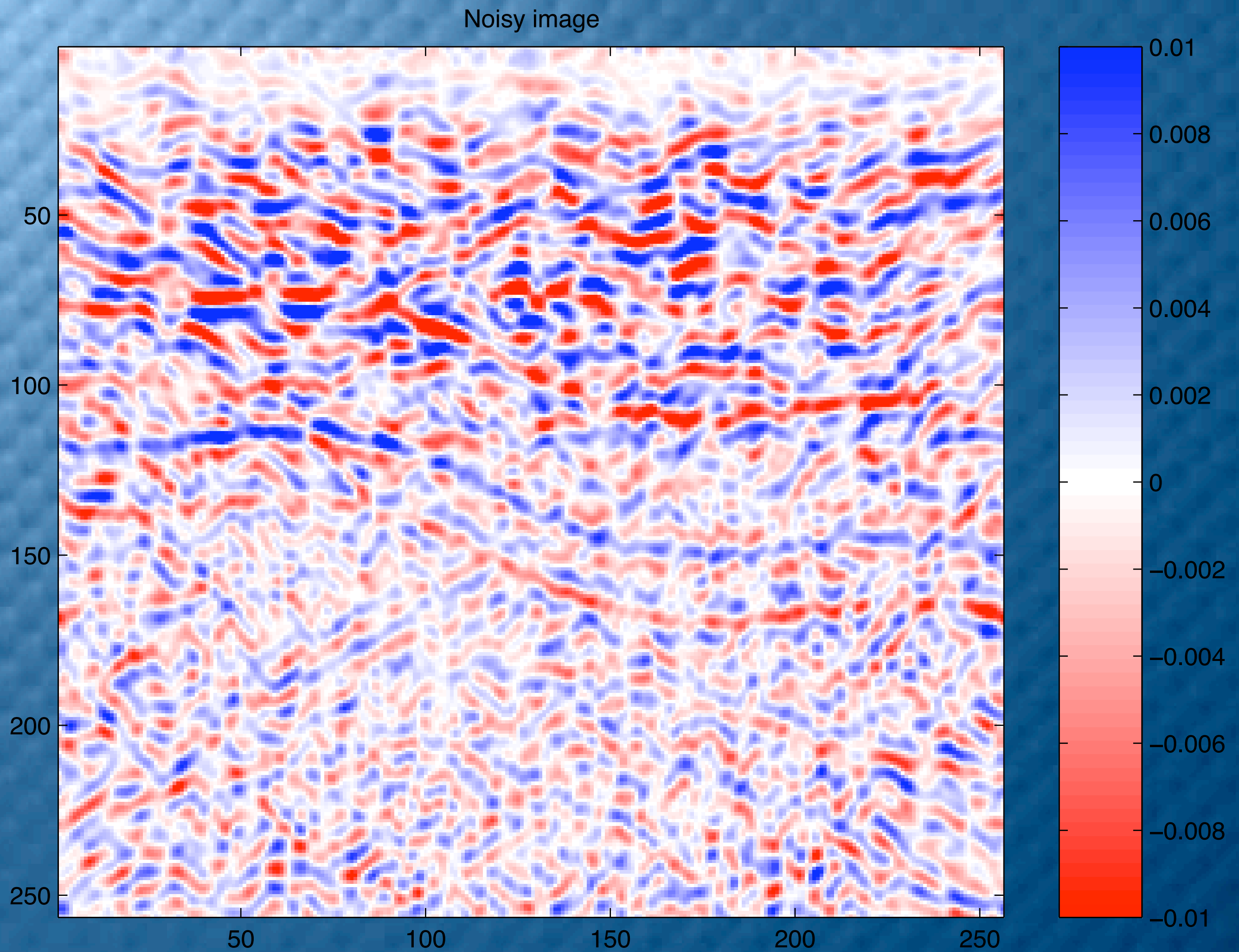
Seismic imaging

We are in the business of

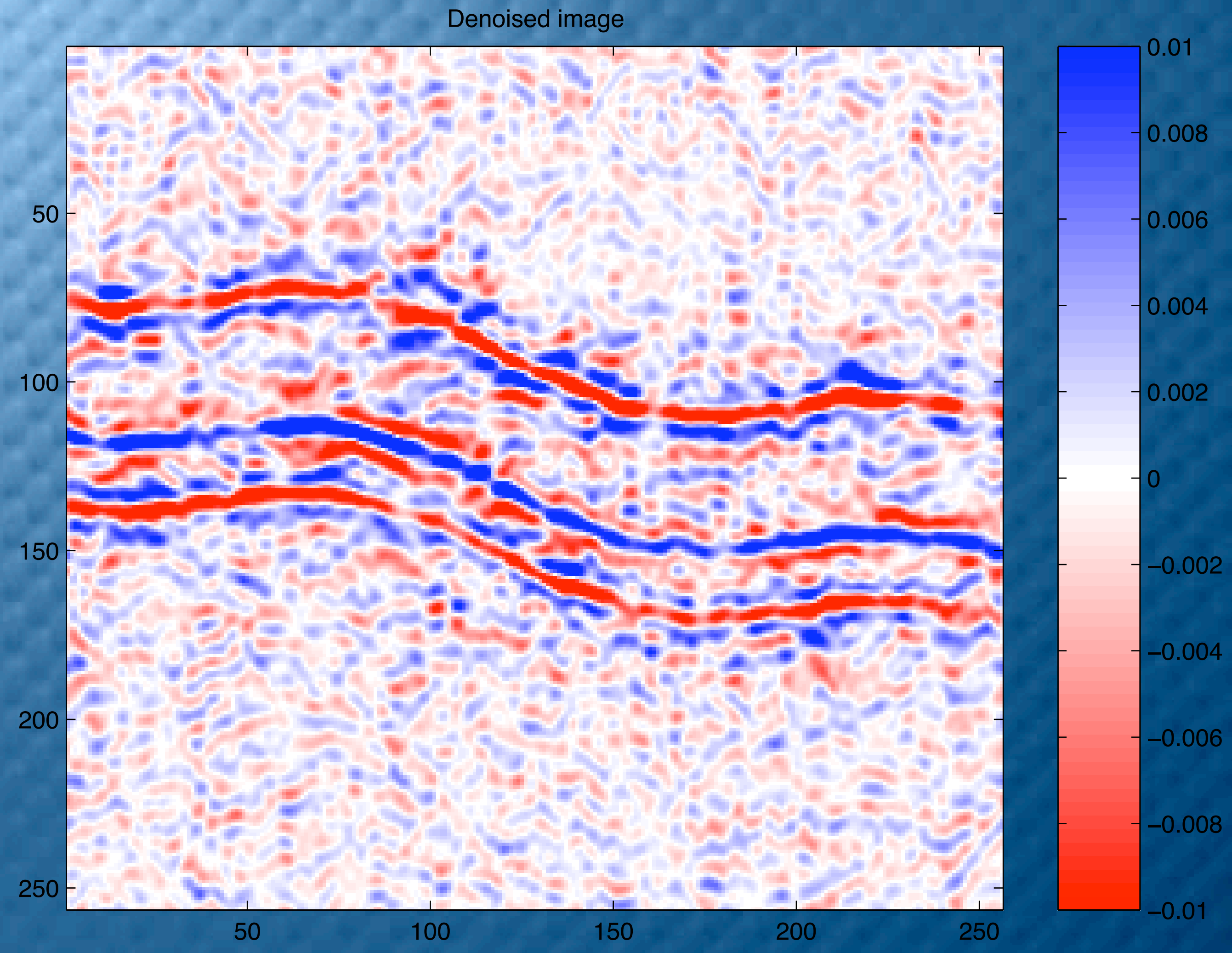
- ***Improving the signal-to-noise ratio (SNR)***
- ***Preserving edges on the model space***
- ***Sparsifying (de)-migration & normal operators***
- **Coming up with *ultimate* preconditioners for *'out-of-the box'* imaging codes**

In the presence of noise ... Lots of it SNR $\leq$ 0***!***

Seismic imaging



‘Optimal’ imaging



Basic idea

Build on the premise that you stand a much better chance of solving an imaging problem when the model is represented optimally ...

- **local in space & angle**
- **sparse**
- **multi-scale and multi-directional**

Well behaved under migration!

Problem

Seismic imaging is an *inverse* problem.

Forward problem:

$$\overbrace{\mathbf{d}}^{\text{data}} = \underbrace{\mathbf{K}}_{\text{scat. oper.}} \overbrace{\mathbf{m}}^{\text{model/refl.}} + \underbrace{\mathbf{n}}_{\text{noise}}$$

invoke *curvelets* to

- **estimate *minimax.***
- ***compress/precondition* operators.**
- **invoke *prior* info (e.g. sparseness).**

Solution

Inverse **problem ↔** ***variational*** **problem.**

Solve (Sacchi)

$$\hat{\mathbf{m}} : \min_{\mathbf{m}} \overbrace{\frac{1}{2}\|\mathbf{d} - \mathbf{K}\mathbf{m}\|_2^2}^{\text{min. data mismatch}} + \underbrace{J(\mathbf{m})}_{\textit{prior}\ \text{info}}$$

Q: Can we use ***multiscale*** **basis functions**

- ***sparse, local*** **&** ***unconditional basis***
- ***well behaved*** **under imaging**
- **preserve the edges**

Inspiration

‘Simple’ (K=I) denoising with hard *threshold:*

$$\hat{\mathbf{m}} = \overbrace{\mathbf{B}^{\dagger}}^{\text{composition}} \underbrace{\Theta^{\mathsf{h}}_{\lambda\Gamma}}_{\text{thres./mute}} \left(\overbrace{\mathbf{B}}^{\text{decomposition}} \mathbf{d} \right)$$

with $\tilde{\mathbf{m}} \triangleq \mathbf{B}\mathbf{m}$ and $\tilde{\mathbf{d}} \triangleq \mathbf{B}\mathbf{d}$ **solves**

$$\hat{\tilde{\mathbf{m}}} \; : \; \min_{\tilde{\mathbf{m}}} \frac{1}{2} \|\tilde{\mathbf{d}} - \tilde{\mathbf{m}}\|_2^2 + \lambda^2 \|\tilde{\mathbf{m}}\|_p$$

Inspiration

Supplement constrained optimization (Candes '02):

$$\hat{\mathbf{m}}: \quad \min_{m} J(\mathbf{m}) \quad \text{s.t.} \quad |\tilde{\mathbf{m}} - \hat{\tilde{\mathbf{m}}}_0|_\mu \leq \mathbf{e}_\mu, \quad \forall \mu$$

with

$$\hat{\tilde{\mathbf{m}}}_0 = \Theta^h_{\boldsymbol{\lambda}\boldsymbol{\Gamma}}\left(\tilde{\mathbf{d}}\right)$$

and

$$\boldsymbol{\Theta}^h_\lambda\left(\tilde{\mathbf{d}}\right) \triangleq \begin{cases} \tilde{\mathbf{d}} & \text{if} \quad |\tilde{\mathbf{d}}| > \lambda \\ 0 & \text{if} \quad |\tilde{\mathbf{d}}| \leq \lambda \end{cases}$$

Minimax estimation

$$\hat{\mathbf{m}} = \mathbf{B}^{\dagger}\Theta_t\left(\mathbf{Bd}\right)$$

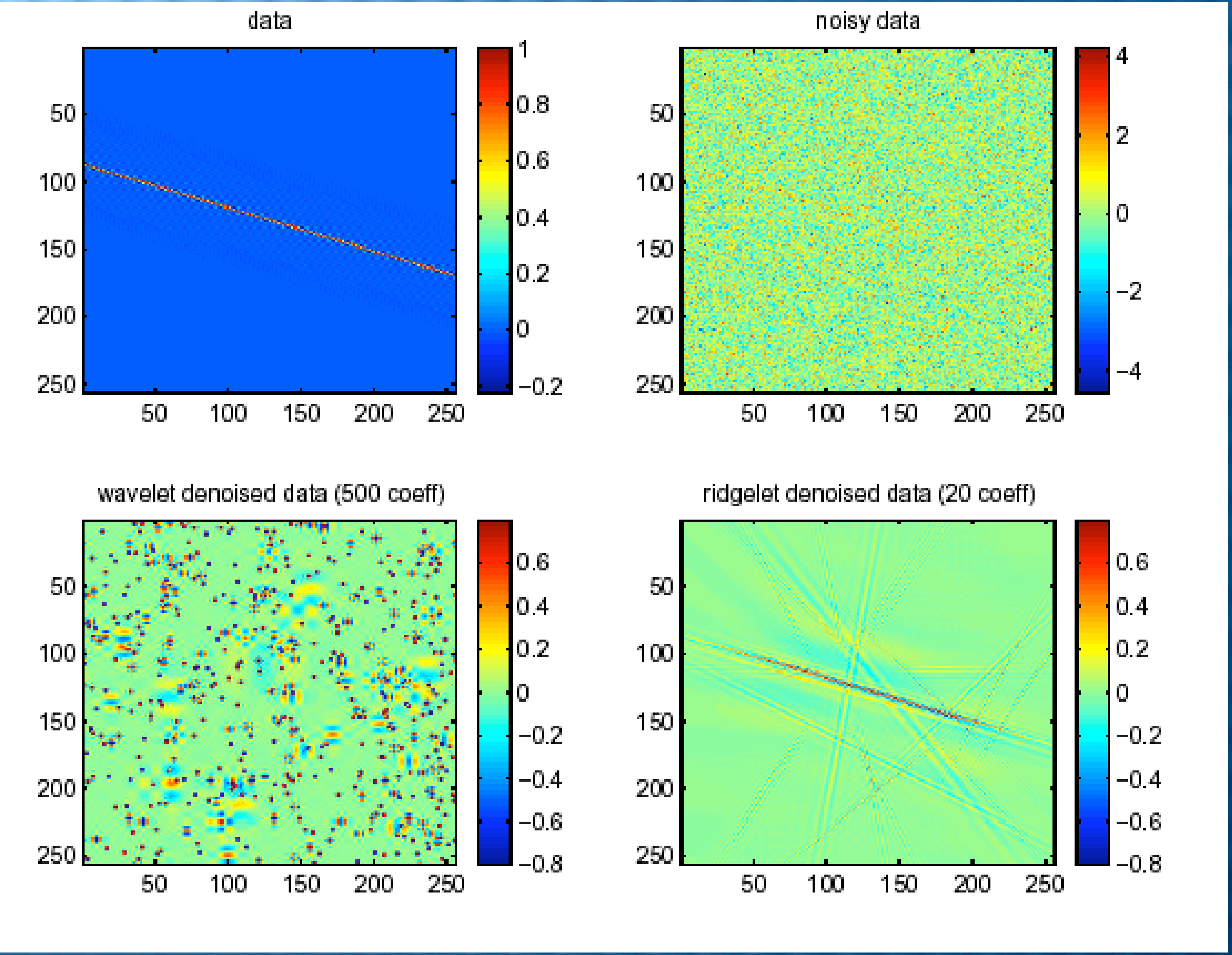
- approximates **minimax**, minimizes max. risk without *prior* **info**
- **Bayes for ‘least favorable’ *prior***
- **preserves edges** $J(\mathbf{m}) \neq \|\mathbf{Lm}\|_2$
- **optimal/unconditional** basis functions

Mallat ‘97
Donoho ‘92

Wavelets

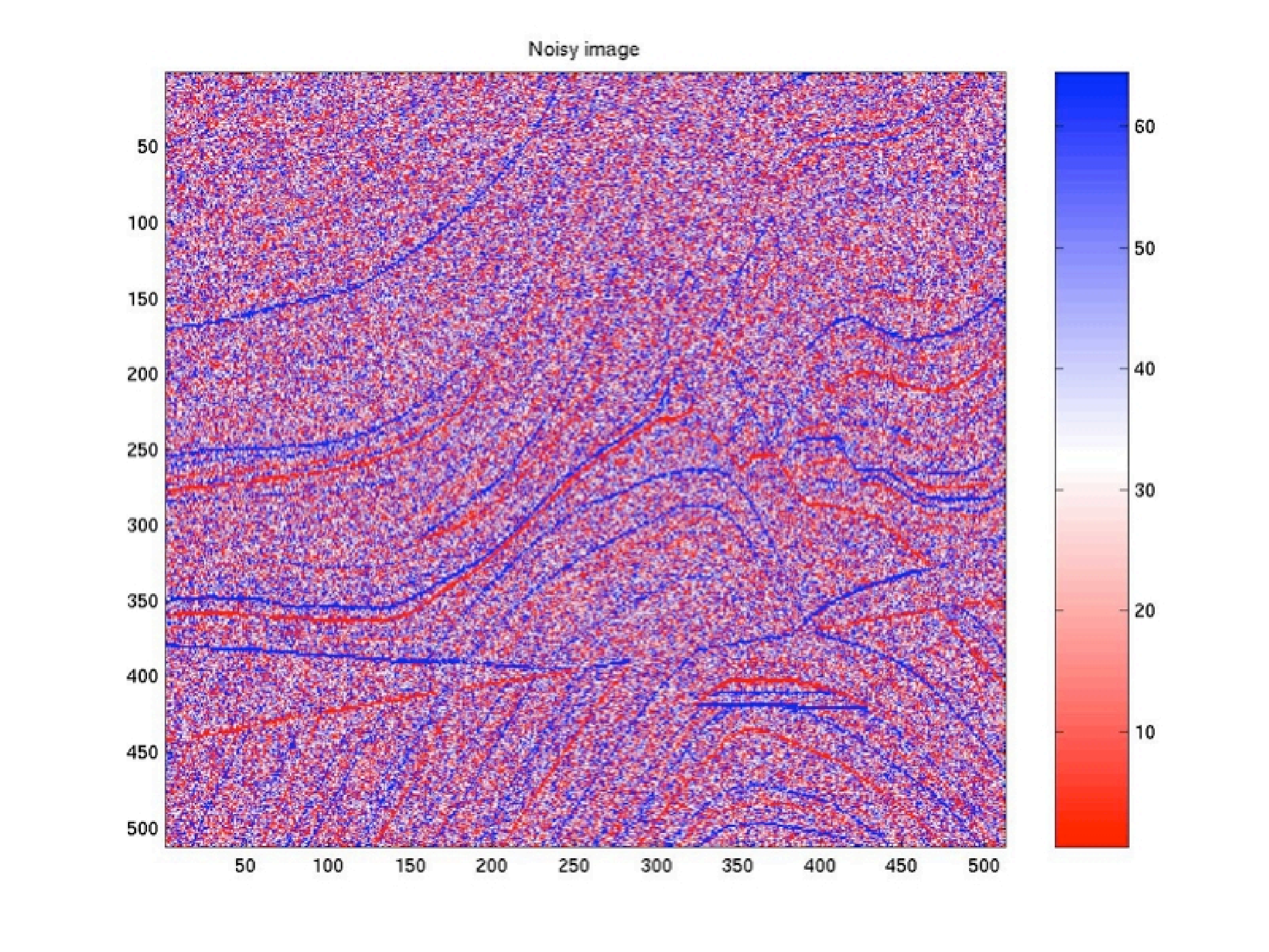
- Represent **piece-wise smooth** functions at “**no**” additional cost.
- Do **not** have to know ***where*** the **singularities** are.
- **Only** good for **point-scatterers** or **horizon/vertically-aligned reflectors.**
- **Do *NOT* compress operators.**
- ***Loose all beneficial properties.***

Directional wavelets

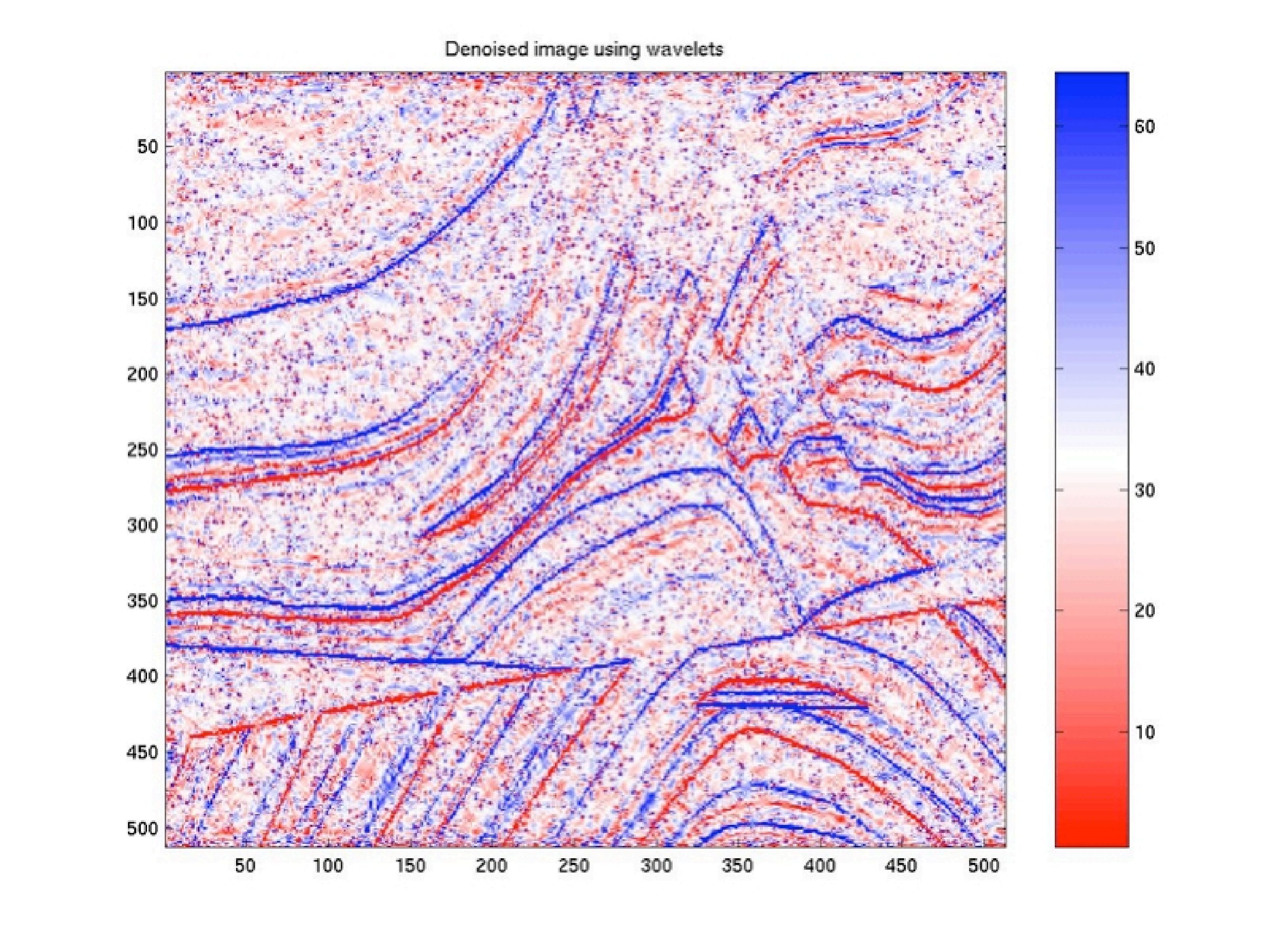


courtesy WaveleLab + Beamlab ‘02

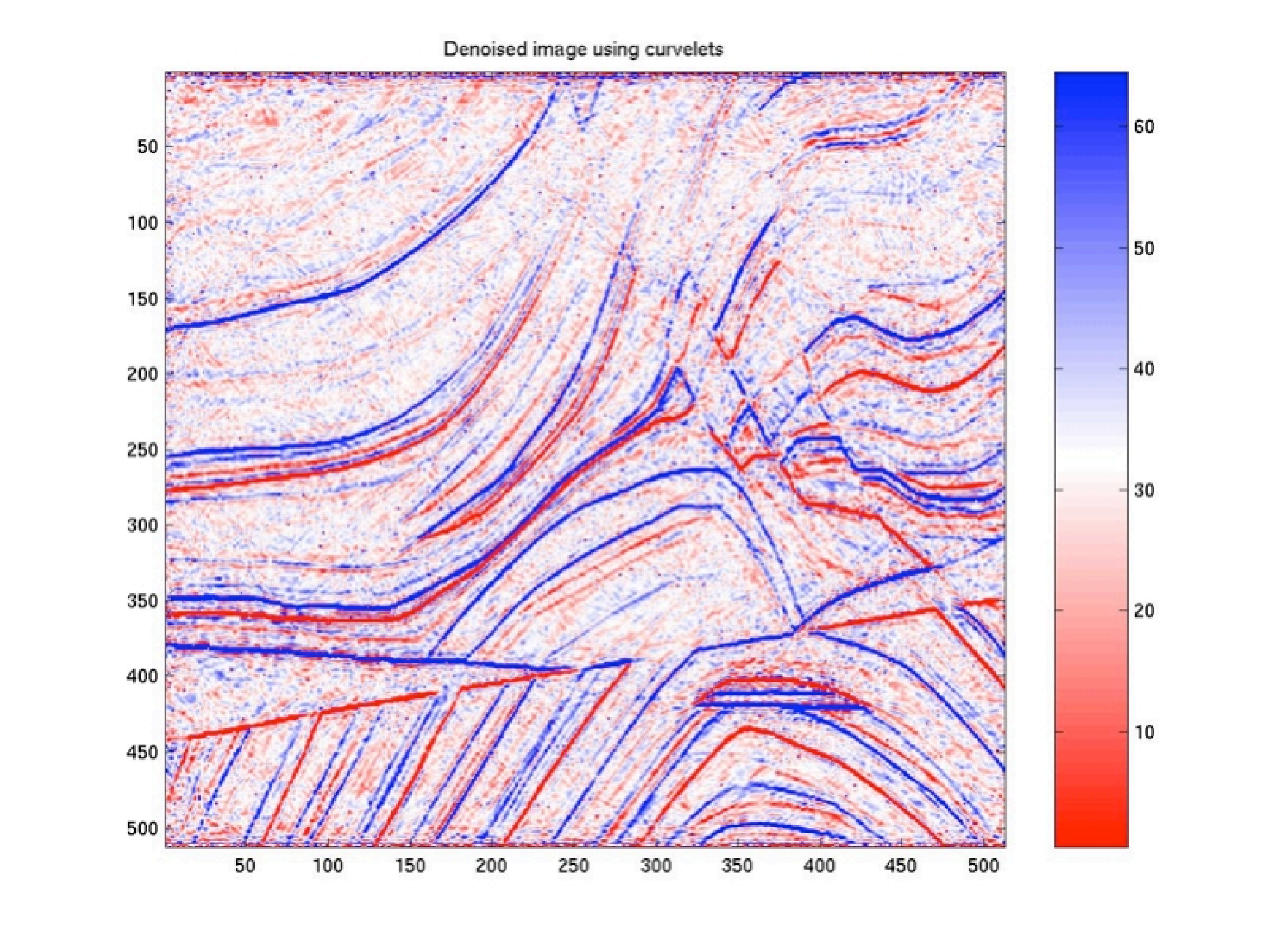
Directional wavelets



Directional wavelets



Directional wavelets



Seismic imaging

Works so well because we exploit

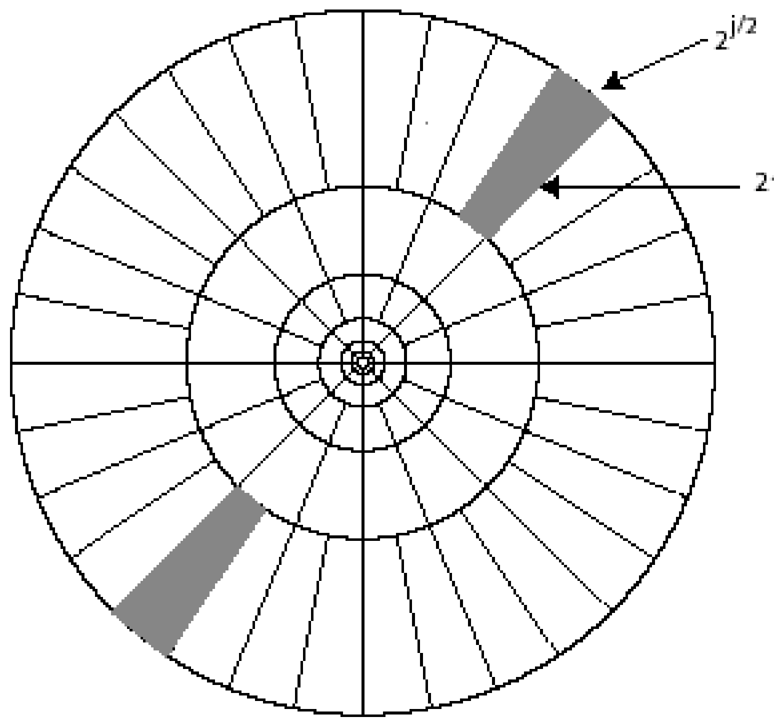
- **continuity *along* reflectors**
- *adaptive* **local** smoothing

Remaining challenges:

- deal with the operator/coloring
- compensate for the normal operator

Curvelet properties

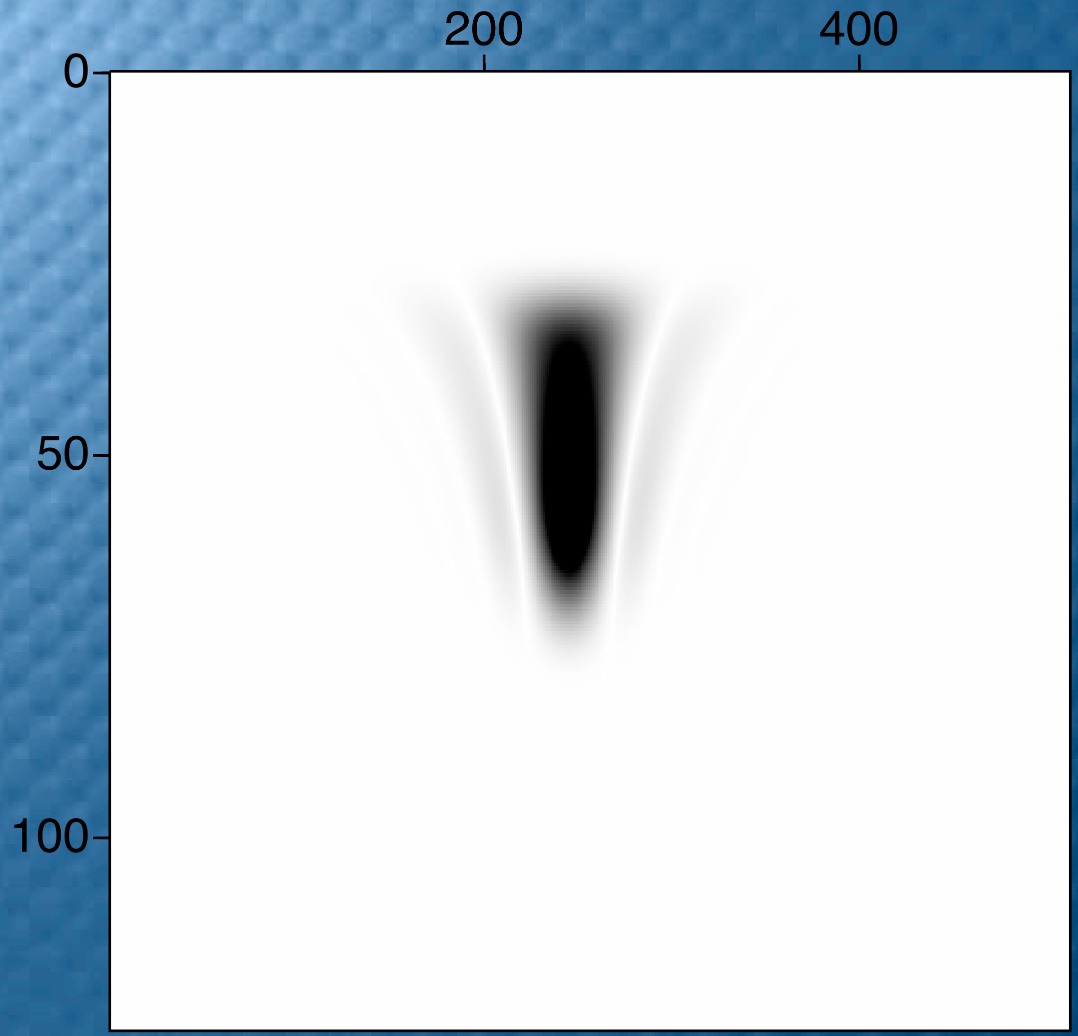
$$W_j = \{\square, \quad 2^j \le |\square| \le 2^{j+1}, \; |\square - \square_J| \le \square \cdot 2^{\lfloor j/2 \rfloor}\}$$



second dyadic partitioning

Stein '93
Candes '02

Curvelet properties



Curvelet in FK-dom□in

Imaging

Q: Extend results to *migration?*

- **deal with operators**
- **reformulate into ‘denoising’ problem**

Curvelets compress FIO’s (also ΨDO’s)

- **exploit compression with Lanczos methods (ultimate preconditioning)**
- **exploit Curvelet properties**
- **define new imaging schemes**

Operators

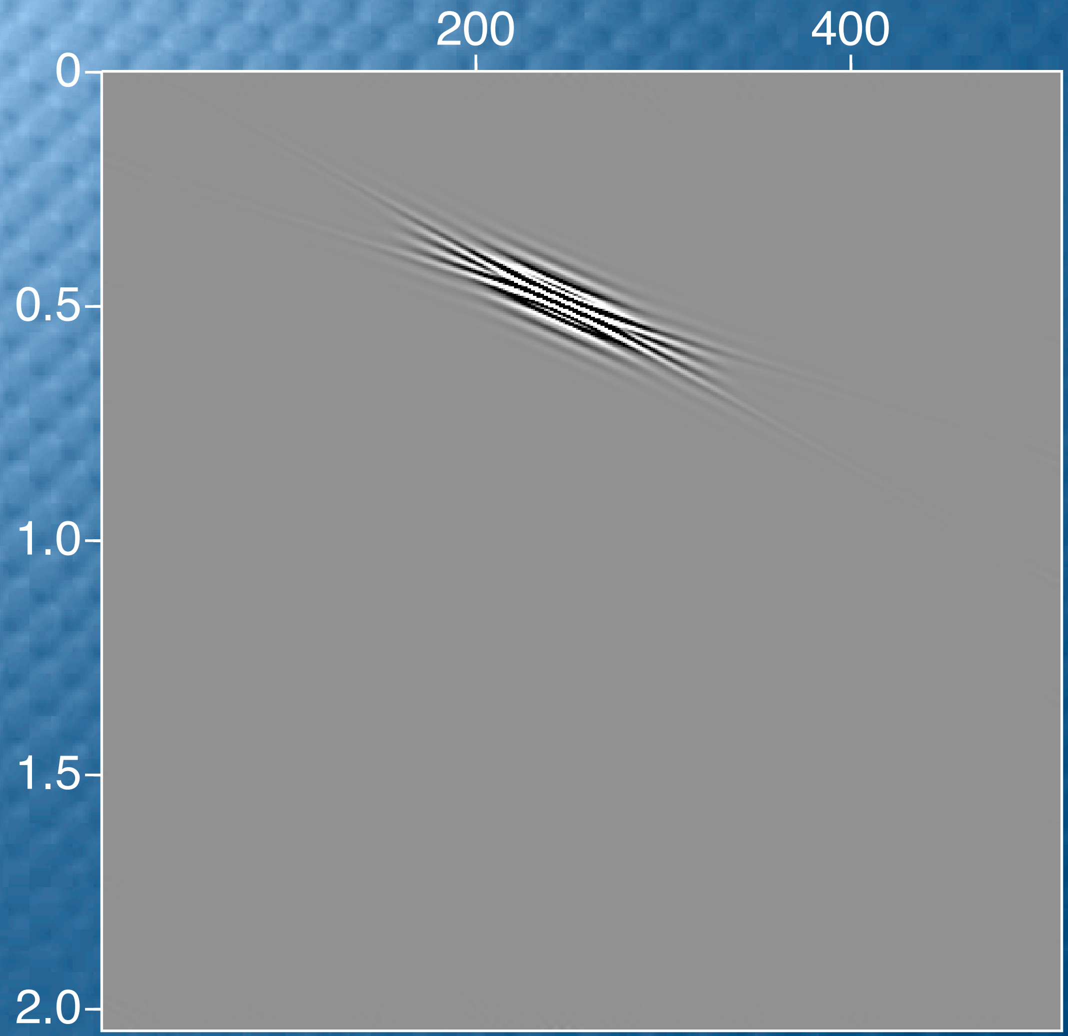
$$\hat{\mathbf{m}} = \overbrace{\left(\mathbf{K}^T\mathbf{K}\right)^{-1}}^{\Psi\mathrm{DO}} \underbrace{\mathbf{K}^T}_{\mathrm{FIO}} \mathbf{d}$$

	ΨDO	FIO	d & m
Wavelets	✗	✗	✗
Curvelets	✔	✔	✔

Theorem from Candes & Demanet ‘04:

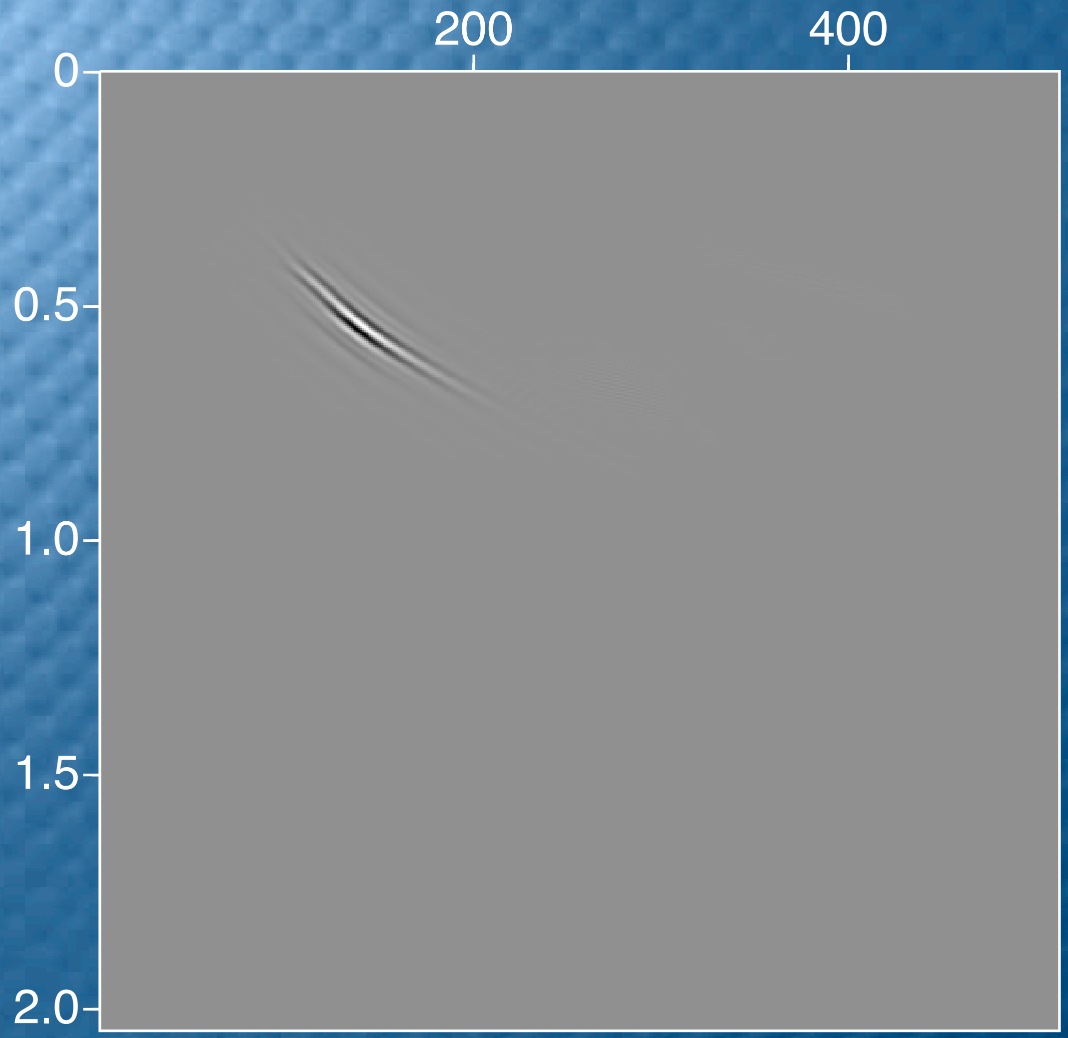
$$\|\mathbf{K}^T\mathbf{d} - \mathbf{K}^T_{\mathrm{trunc.}}\mathbf{d}\|_2 \leq C(\#\,\mathrm{per\ col.})^{-M}\,\mathrm{for\ each}\ M$$

Operators



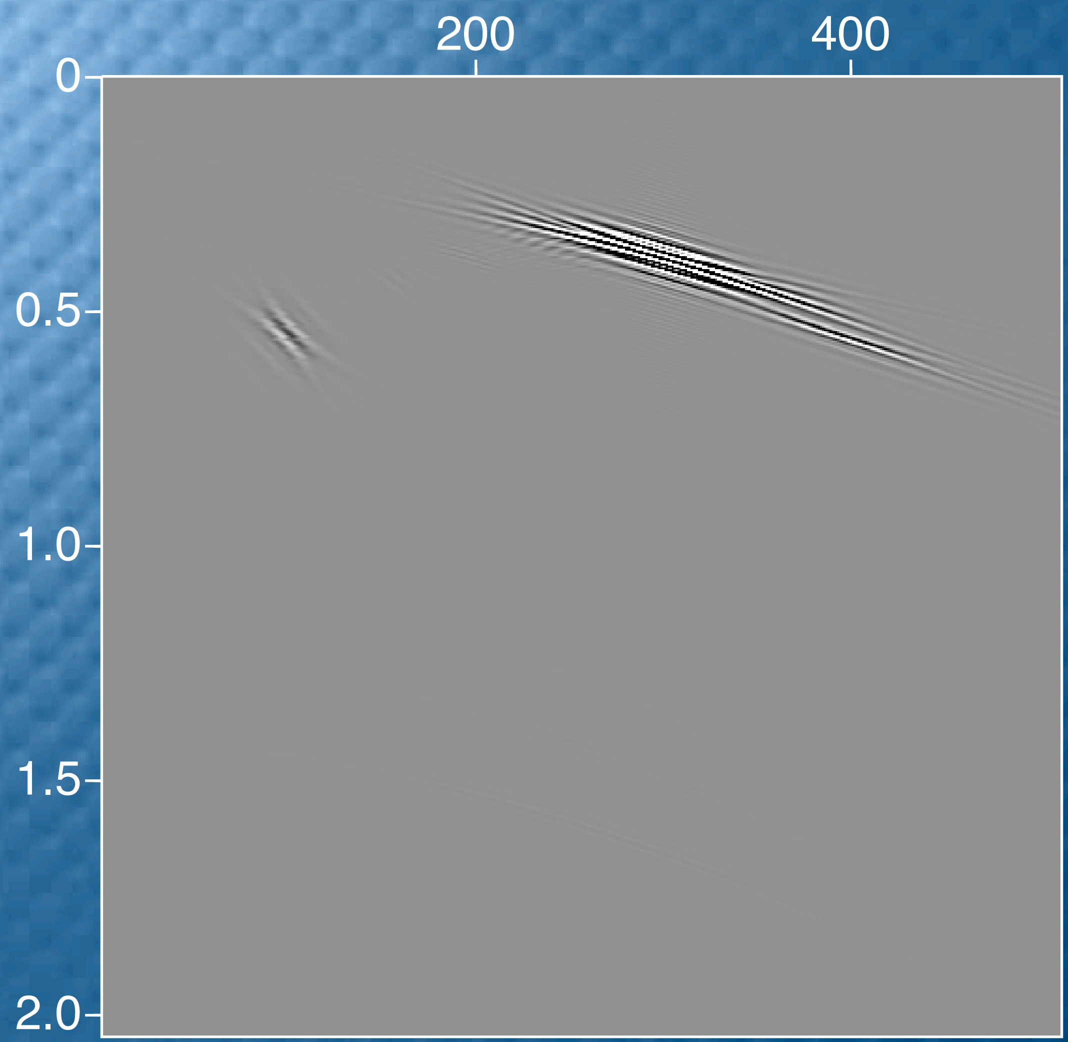
A Curvelet

Operators



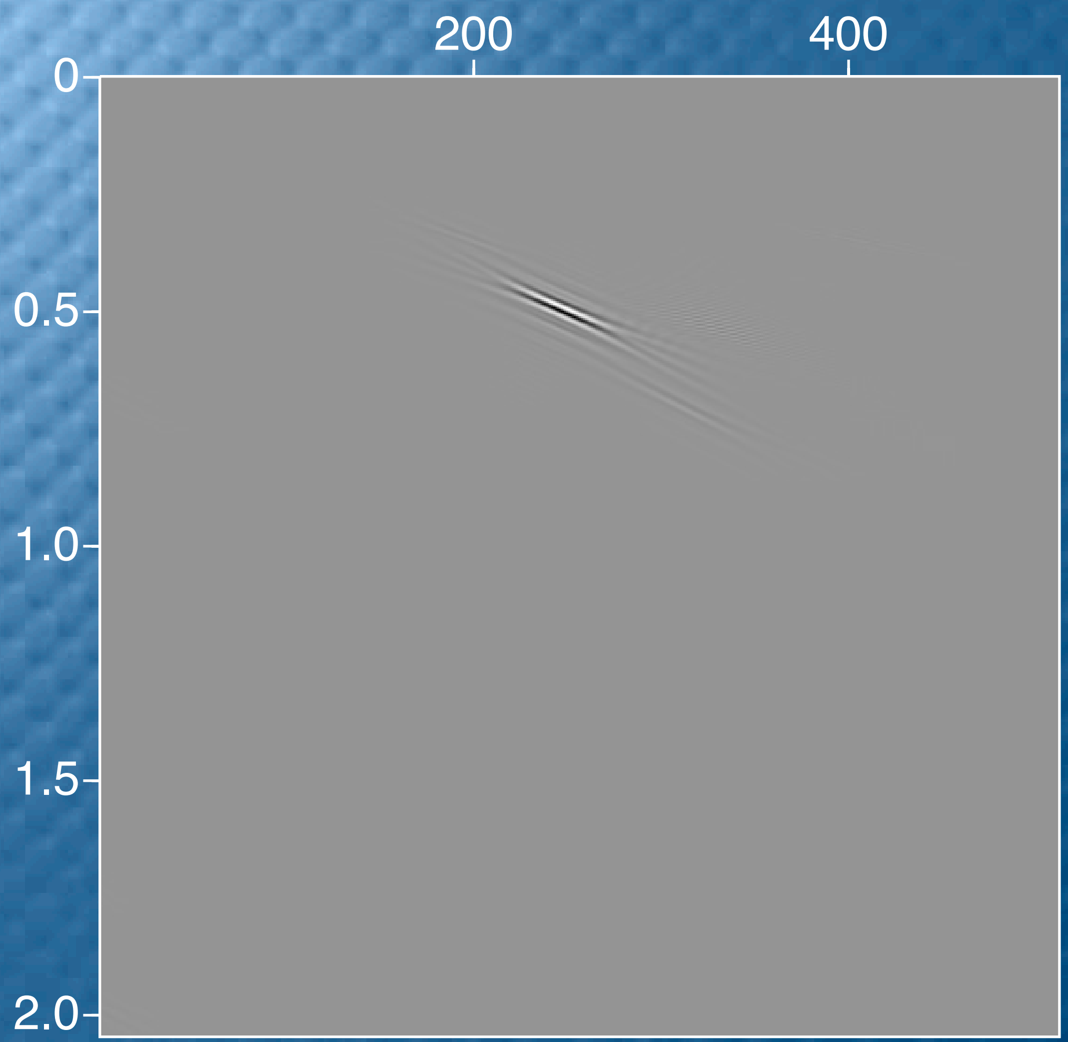
Migr□ted Curvelet

Operators



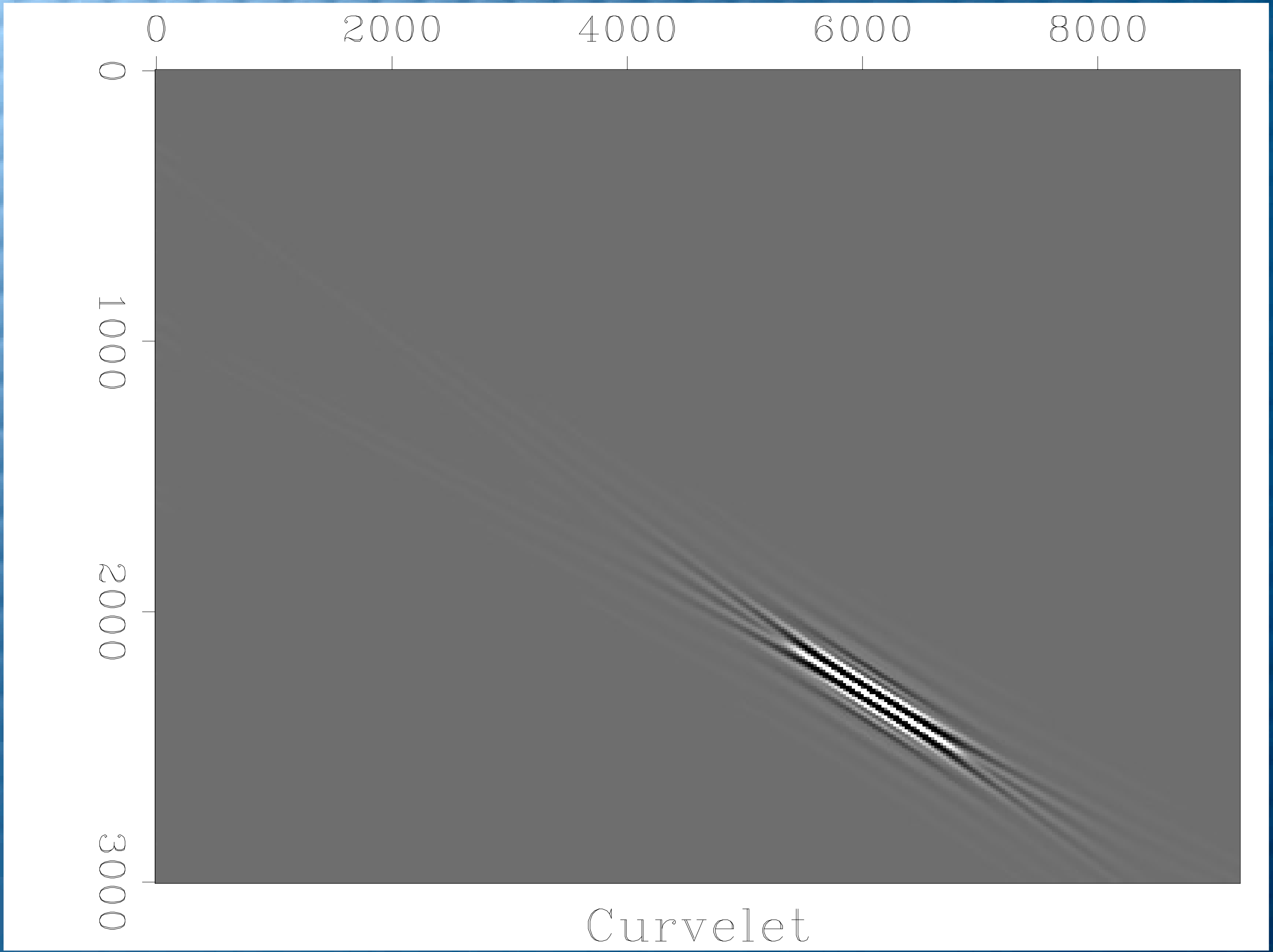
Demigr□ted Curvelet

Operators

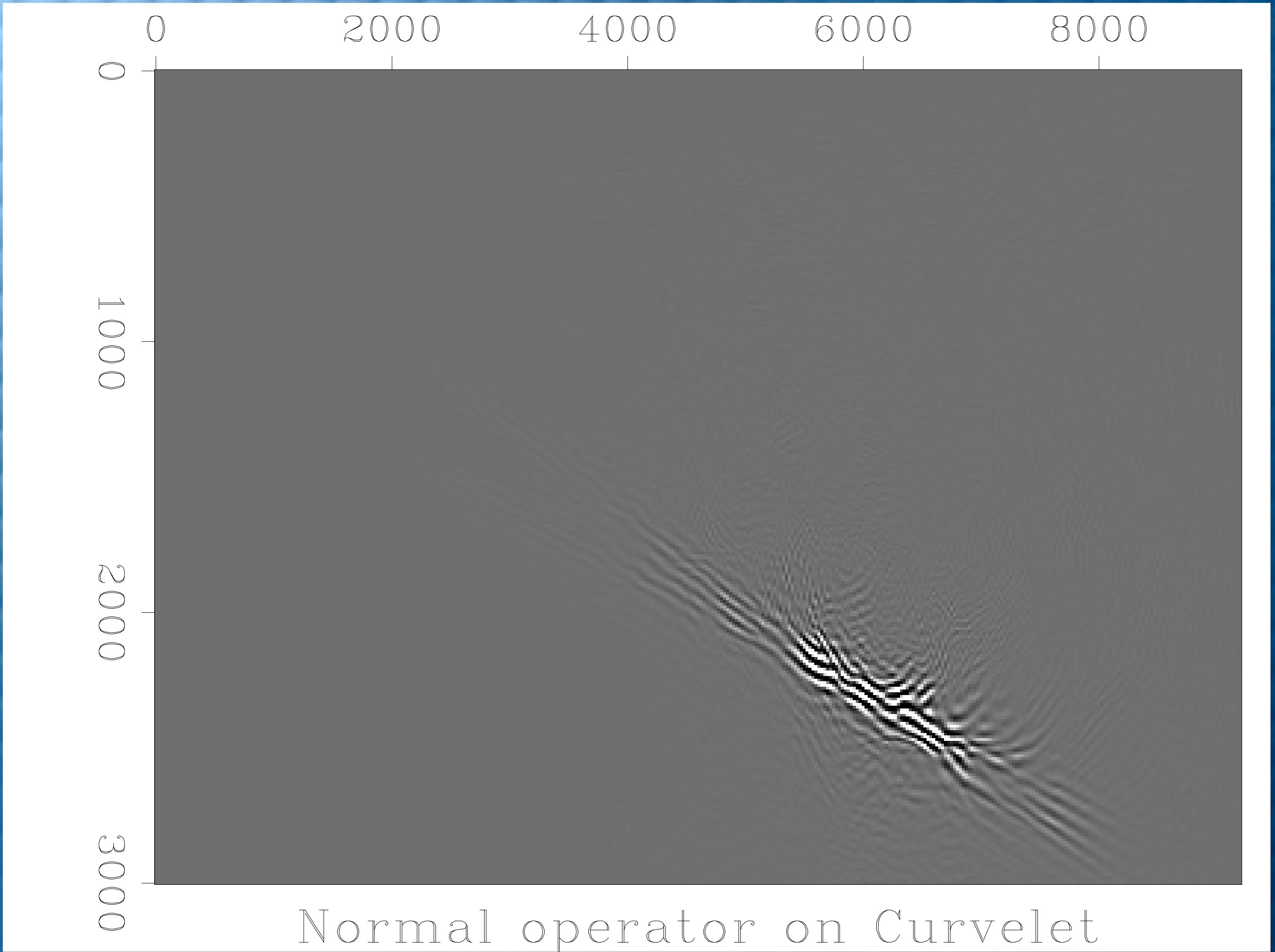


Demigr□ted migr□ted Curvelet

Operators

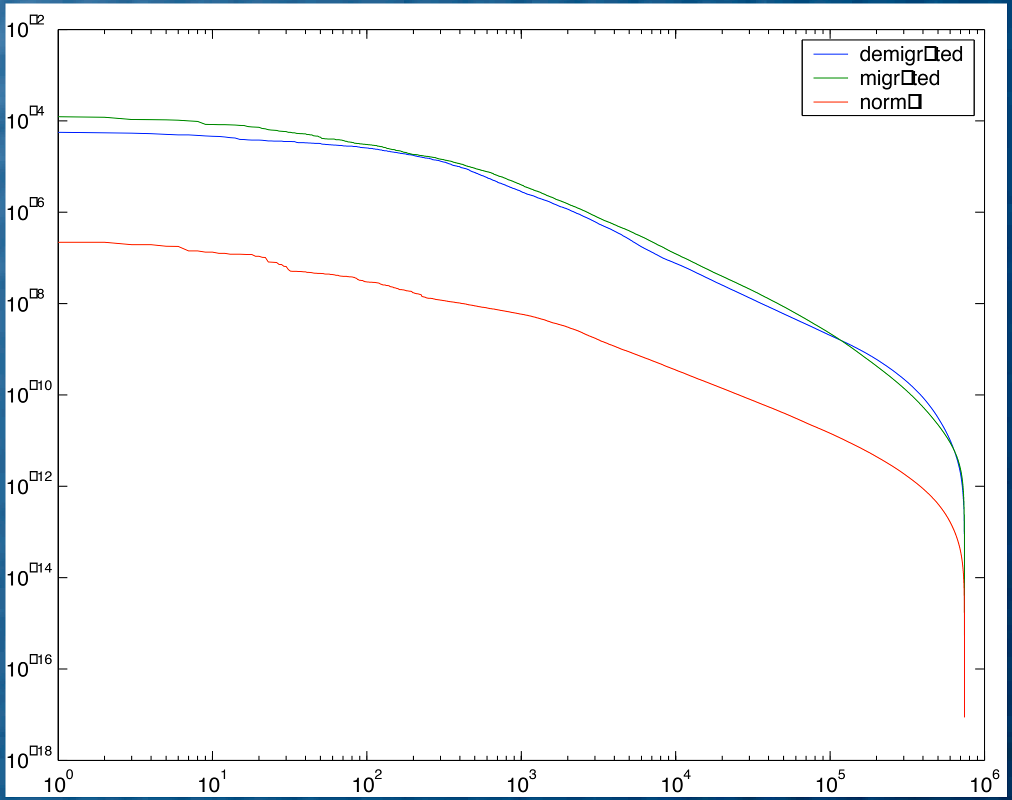


Operators


0
2000
4000
6000
8000
0
1000
2000
3000
Normal operator on Curvelet

Operators

- **Curvelets remain curvelet-like**
- **Compress the operators**
- **Almost diagonalize normal operator**



In particular

$$\mathbf{B}\mathbf{K}^T\mathbf{K}\mathbf{B}^T \approx \mathrm{diag}\left(\mathrm{diag}\left(\mathbf{B}\mathbf{K}^T\mathbf{K}\mathbf{B}^T\right)\right) \triangleq \Gamma^2$$

Preconditioning

Reformulate into preconditioned *normal* equations:

$$\mathbf{F}^T\mathbf{d} = \mathbf{F}^T\mathbf{F}\mathbf{x} + \mathbf{F}^T\mathbf{n}_{\text{old}}$$

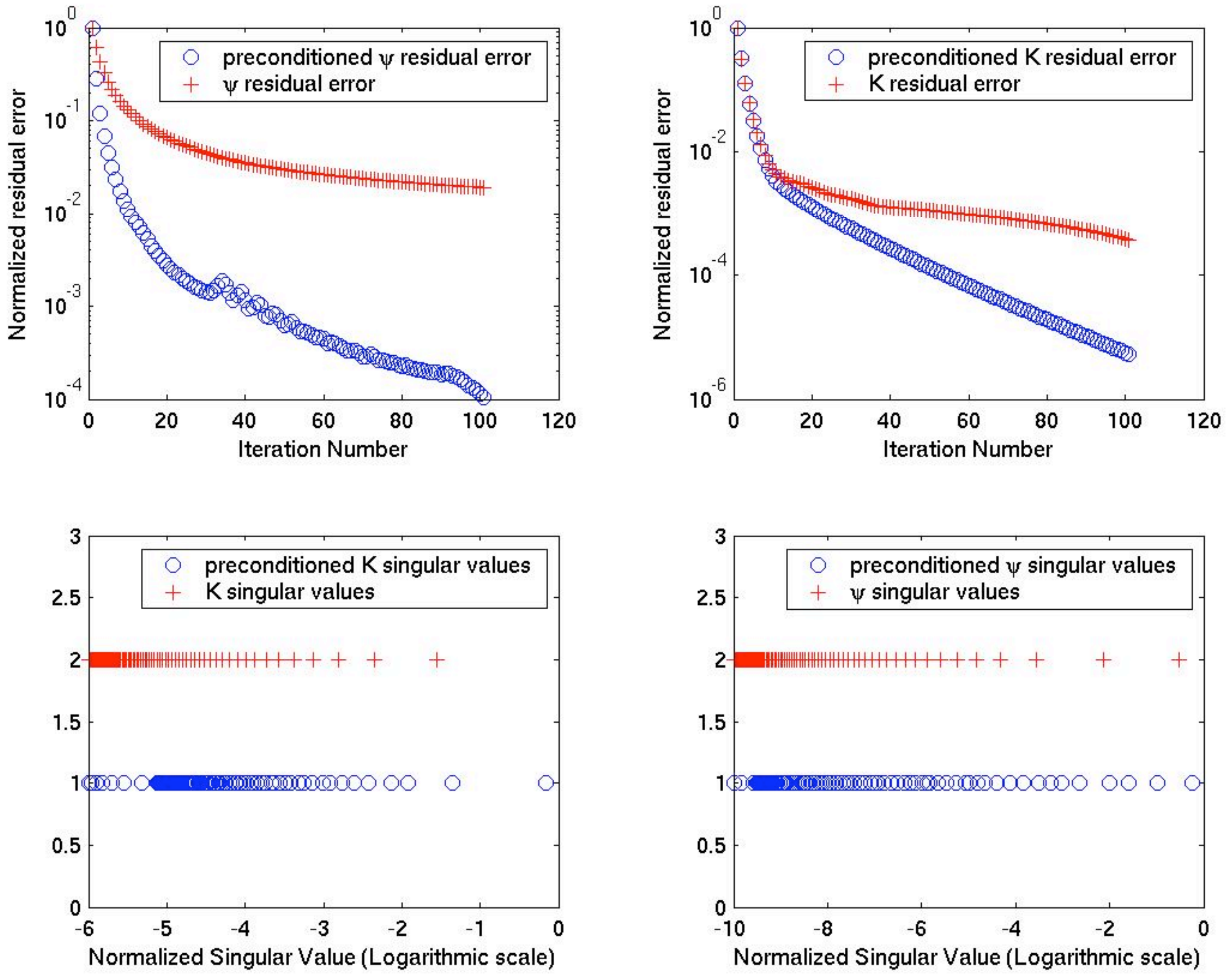
with

$$\mathbf{F} = \mathbf{K}\mathbf{P},\ \mathbf{x} = \mathbf{P}^T\mathbf{m} \text{ and } \mathbf{P} = \mathbf{C}^T\mathbf{\Gamma}^{-1}$$

yielding

$$\mathbf{y} = \overbrace{\mathbf{A}}^{\approx\mathbf{I}}\mathbf{x} + \underbrace{\mathbf{n}}_{\text{'white'}}$$

Preconditioning



Estimation

‘Ignore’ operator ($\mathbf{A} \approx \mathbf{I}$):

$$\hat{\mathbf{x}}_0 = \boldsymbol{\Theta}_\lambda\left(\mathbf{y}\right)$$

equivalent to

$$\hat{\mathbf{m}}_0 = \mathbf{B}^\dagger \left(\boldsymbol{\Gamma}^2\right)^\dagger \Theta_{\lambda\Gamma}\left(\mathbf{B}\mathbf{K}^T\mathbf{d}\right)$$

- **approx. compensated for the *normal* operator**
- ***minimax estimator* brings us into *convex***

Estimation

Impose *prior* info *via* constrained opt.

$$\hat{\mathbf{m}} : \quad \min_{\mathbf{m}} J(\mathbf{m}) \quad \text{s.t.} \quad |\mathbf{x} - \hat{\mathbf{x}}_0|_\mu \leq \mathbf{e}_\mu \quad \forall \mu$$

with

$$\mathbf{e}_\mu = \begin{cases} \mathbf{I}_\mu & \text{if} \quad |\hat{\mathbf{x}}_0|_\mu \quad \geq \quad |\boldsymbol{\lambda}\mathbf{I}|_\mu \\ \boldsymbol{\lambda}\mathbf{I}_\mu & \text{if} \quad |\hat{\mathbf{x}}_0|_\mu \quad < \quad |\boldsymbol{\lambda}\mathbf{I}|_\mu \end{cases}$$

and

$$J(\mathbf{m}) = \|\mathbf{m}\|_1$$

Estimation

Include approx. normal operator

$$\hat{\mathbf{m}} : \quad \min_{\mathbf{m}} J(\mathbf{m}) \quad \text{s.t.} \quad |\mathbf{A}_{\text{lan}}\mathbf{x} - \hat{\mathbf{x}}_0|_\mu \leq \mathbf{e}_\mu \quad \forall \mu$$

with compressed operator

$$\mathbf{A}_{\text{lan}} = \mathbf{Q}\mathbf{T}_k\mathbf{Q}_k^T$$

and

$$\mathbf{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & \ddots & \ddots & \beta_{k-1} \\ 0 & \cdots & & \beta_{k-1} & \alpha_k \end{pmatrix}.$$

Estimation

Covariance operator:

$$\mathbf{C}_{\tilde{\mathbf{n}}} = \mathbf{E}\{\tilde{\mathrm{n}}\tilde{\mathrm{n}}^{\mathbf{T}}\} = \mathbf{B}\mathbf{K}^T\mathbf{K}\mathbf{B}^T$$

with $\tilde{\mathrm{n}} \triangleq \mathrm{BK}^T$

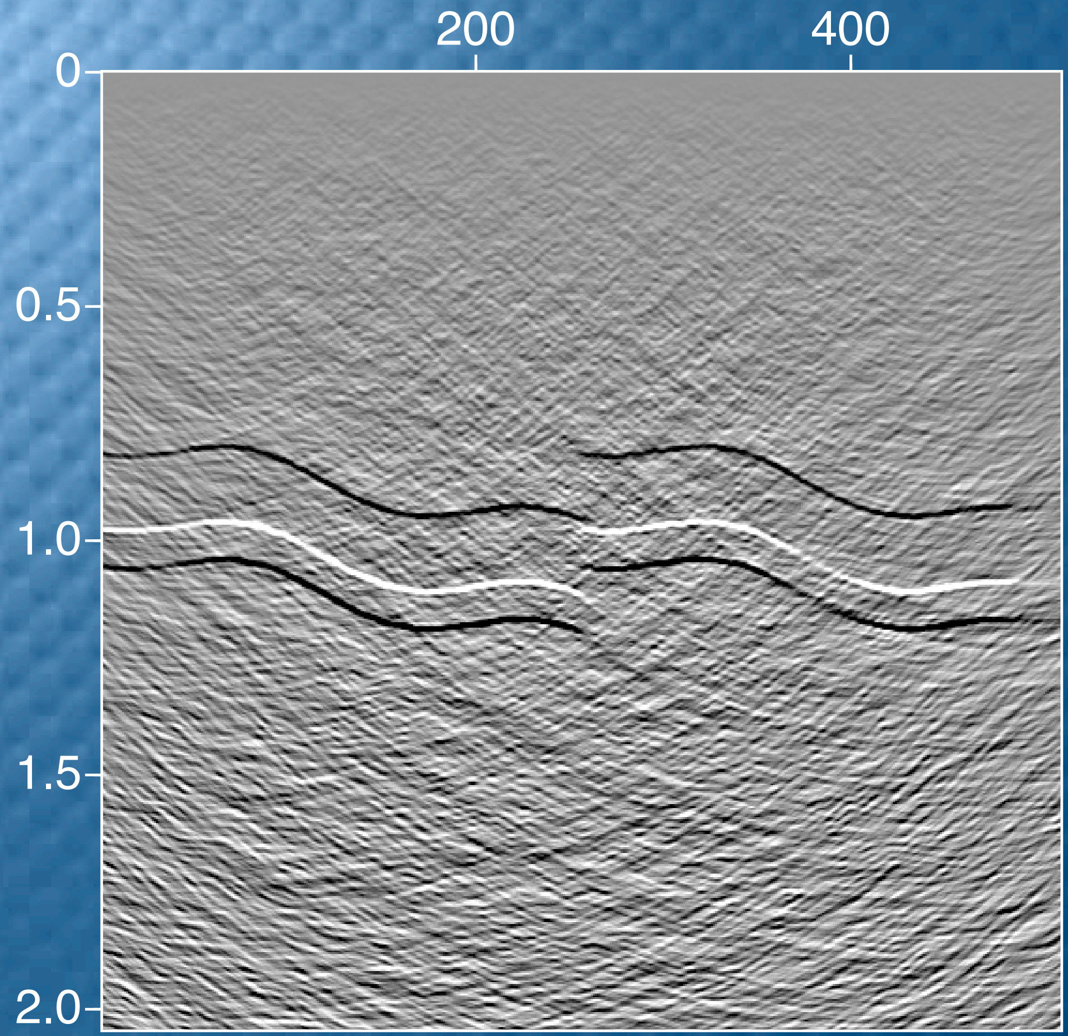
Monte-Carlo sample:

$$\Gamma^2 \triangleq \mathrm{diag}\left(\mathrm{diag}\left(\mathbf{C}_{\tilde{\mathbf{n}}}\right)\right) \approx \frac{1}{N}\sum_{k=1}^{N}\tilde{\mathbf{n}}_k^2$$

Examples

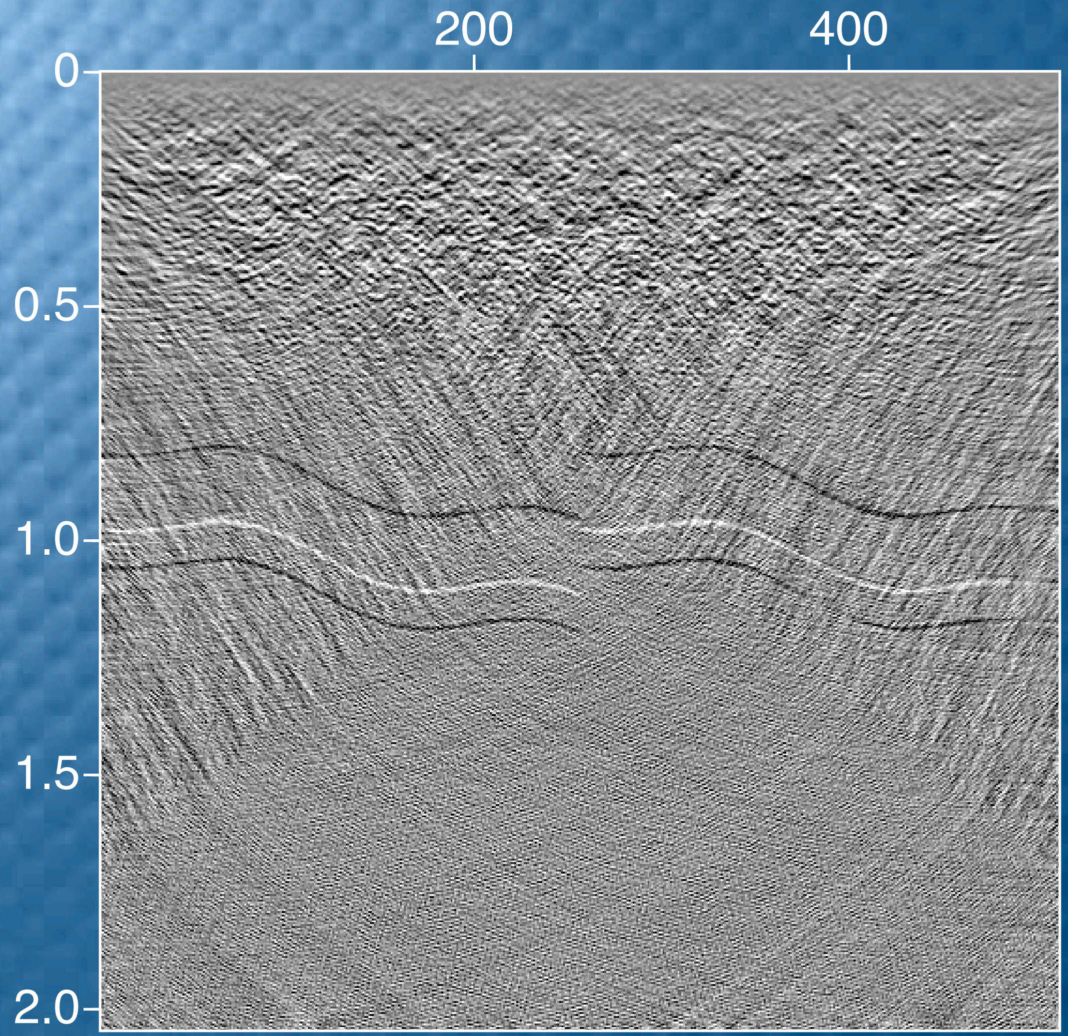
- Common-offset Kirchoff migration
  - constant velocity model
  - simple reflectivity
- Post-stack ‘wave-equation’ migration
  - Marmousi model
  - complicated reflectivity

Examples



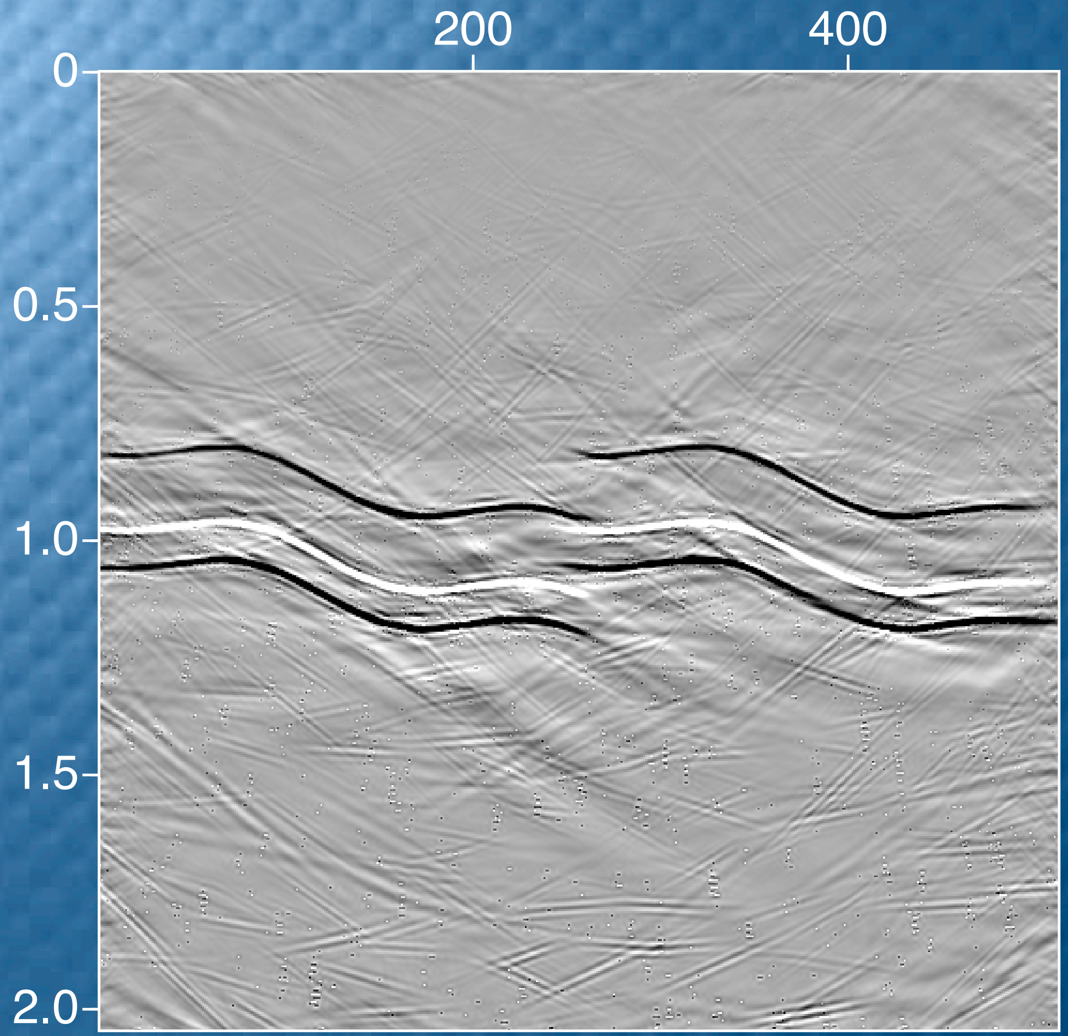
Noisy Image

Examples



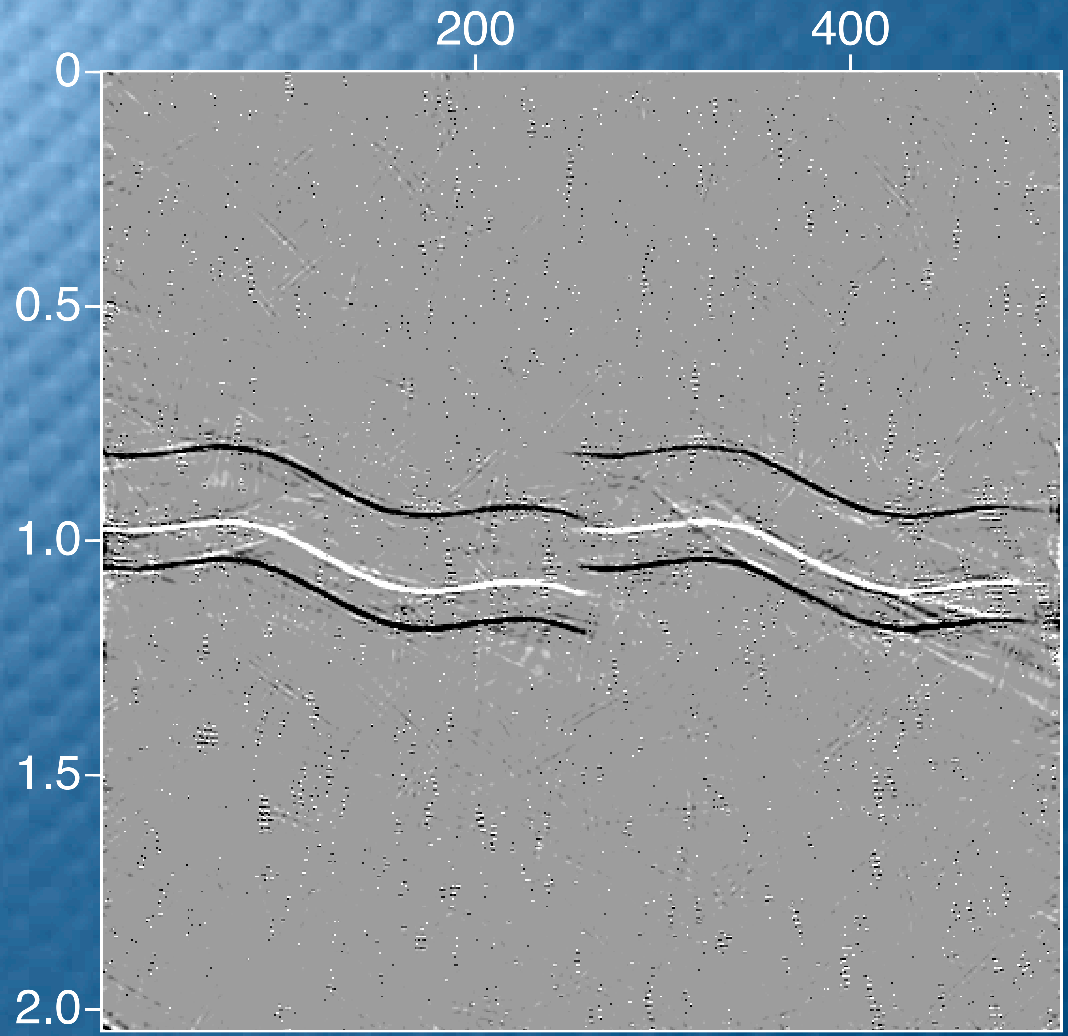
Least-squares migrated Image

Examples



Denoised after Thresholding

Examples

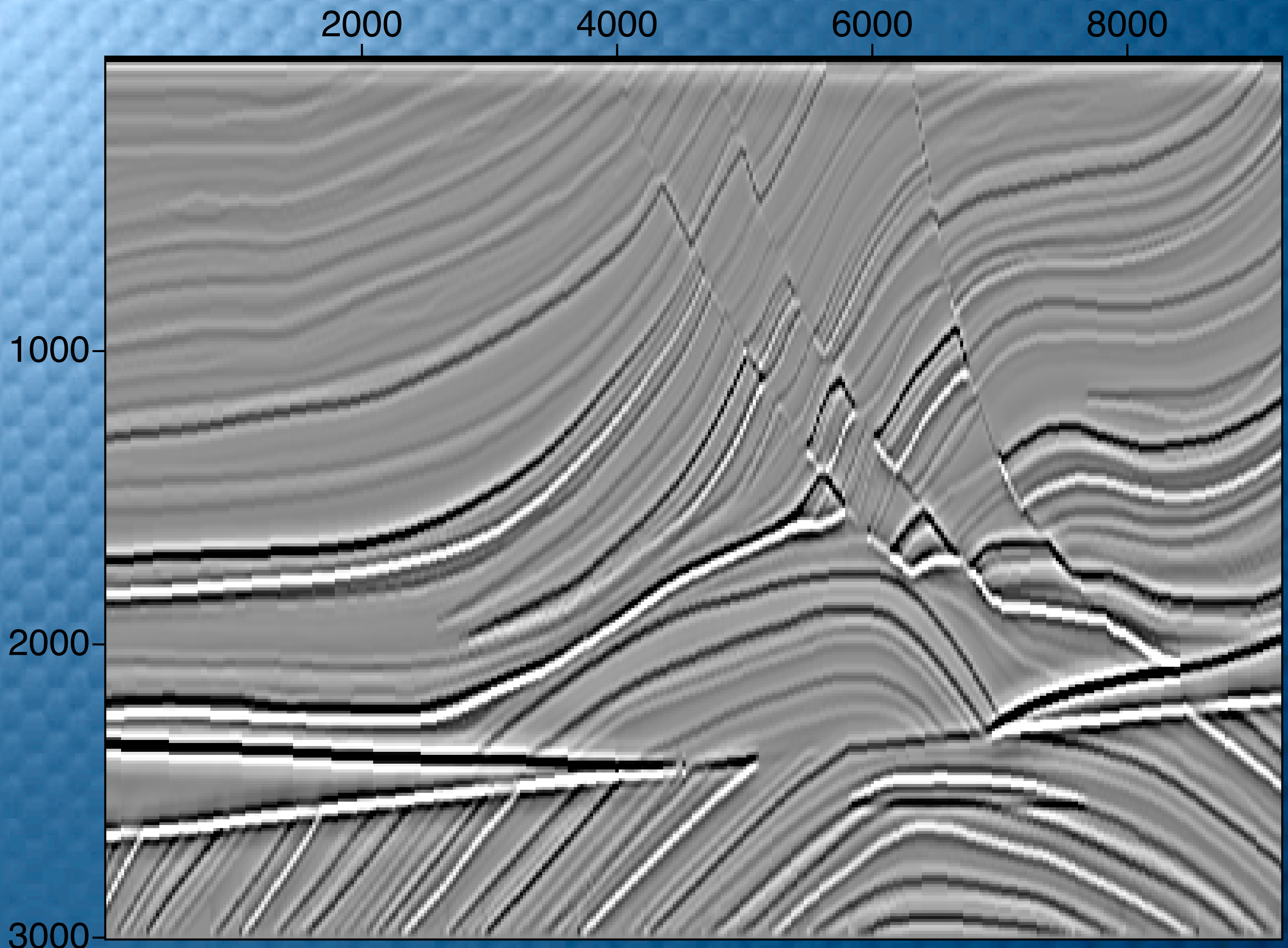


Constrained Optimization

Observations

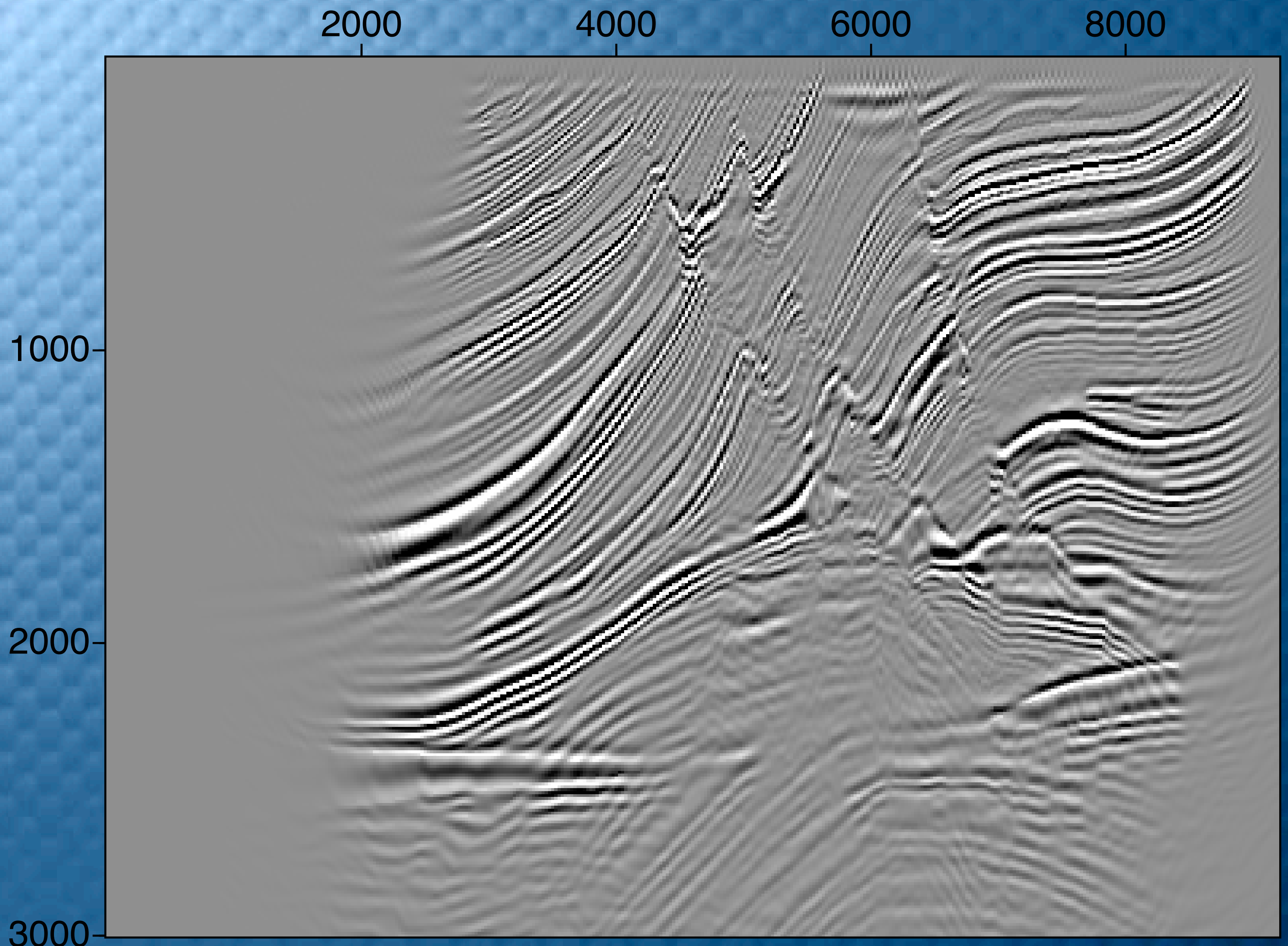
- **Iterative non-regularized Least-squares imaging ‘fits the noise’.**
- **Thresholding preserves the edges.**
- **Normal-operator correction restores the amplitudes.**
- **Constrained optimization removes the artifacts.**
- **Spikes remain due to** L^1

Examples



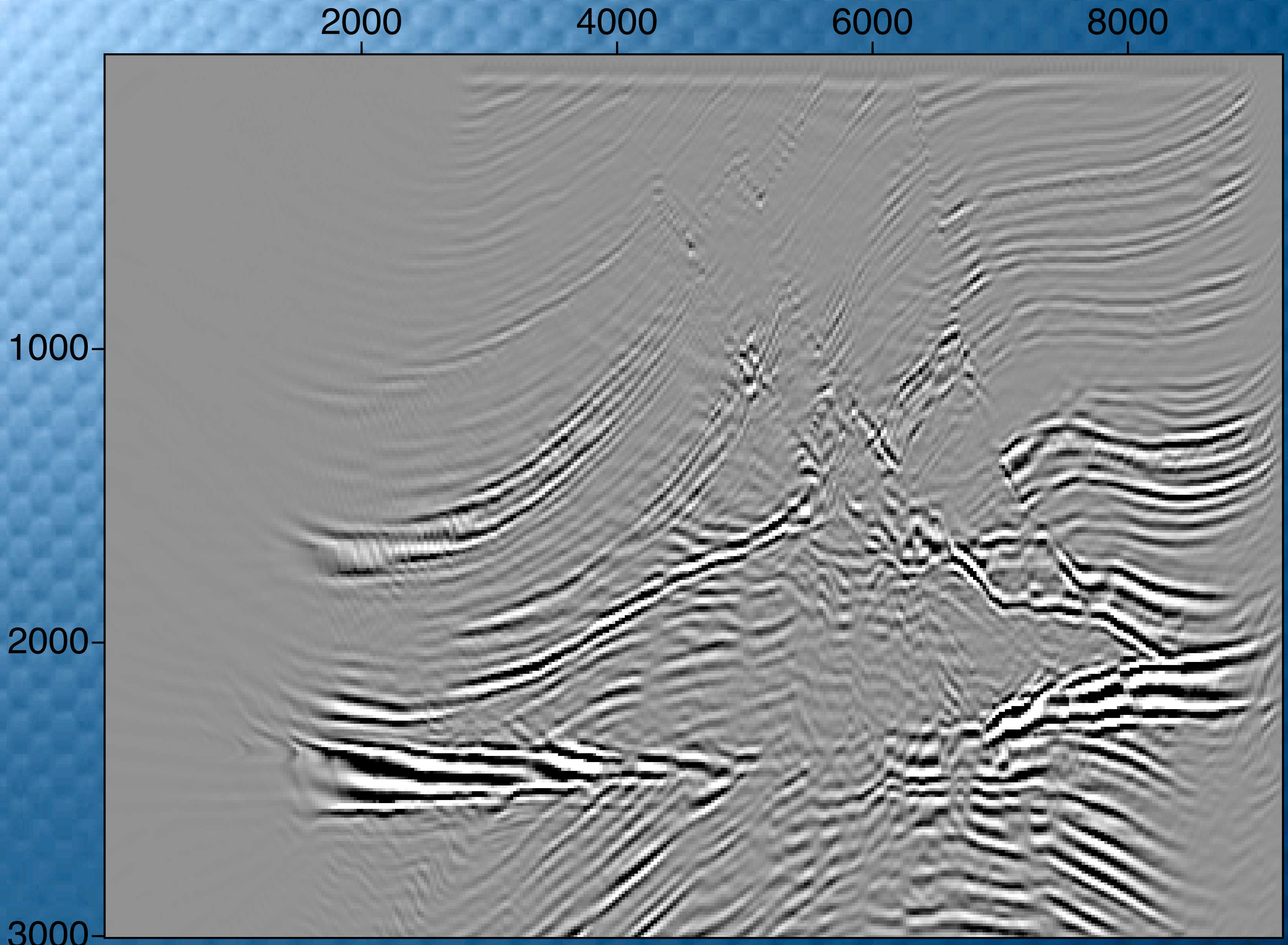
True model

Examples



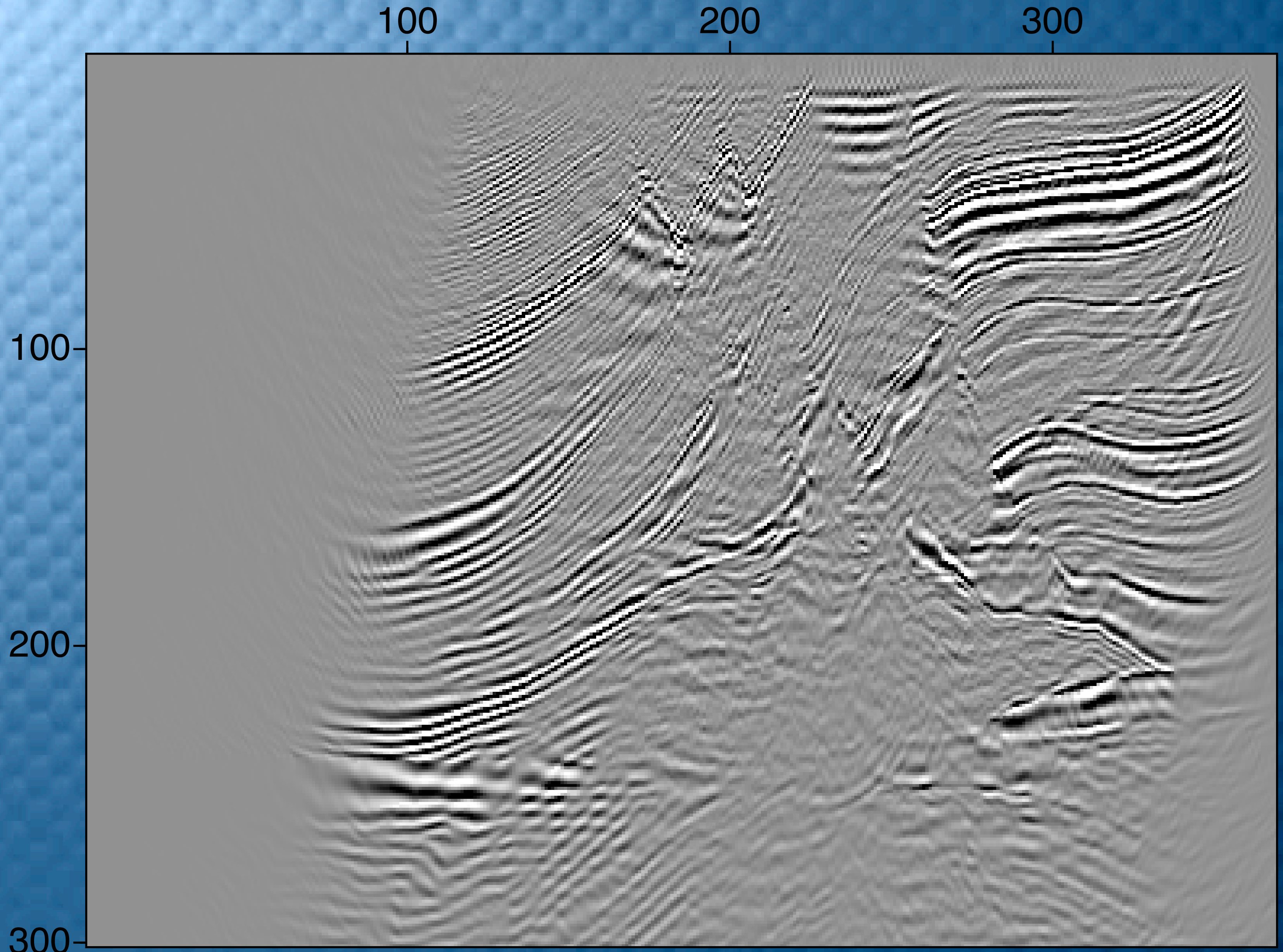
Noise-free image

Examples



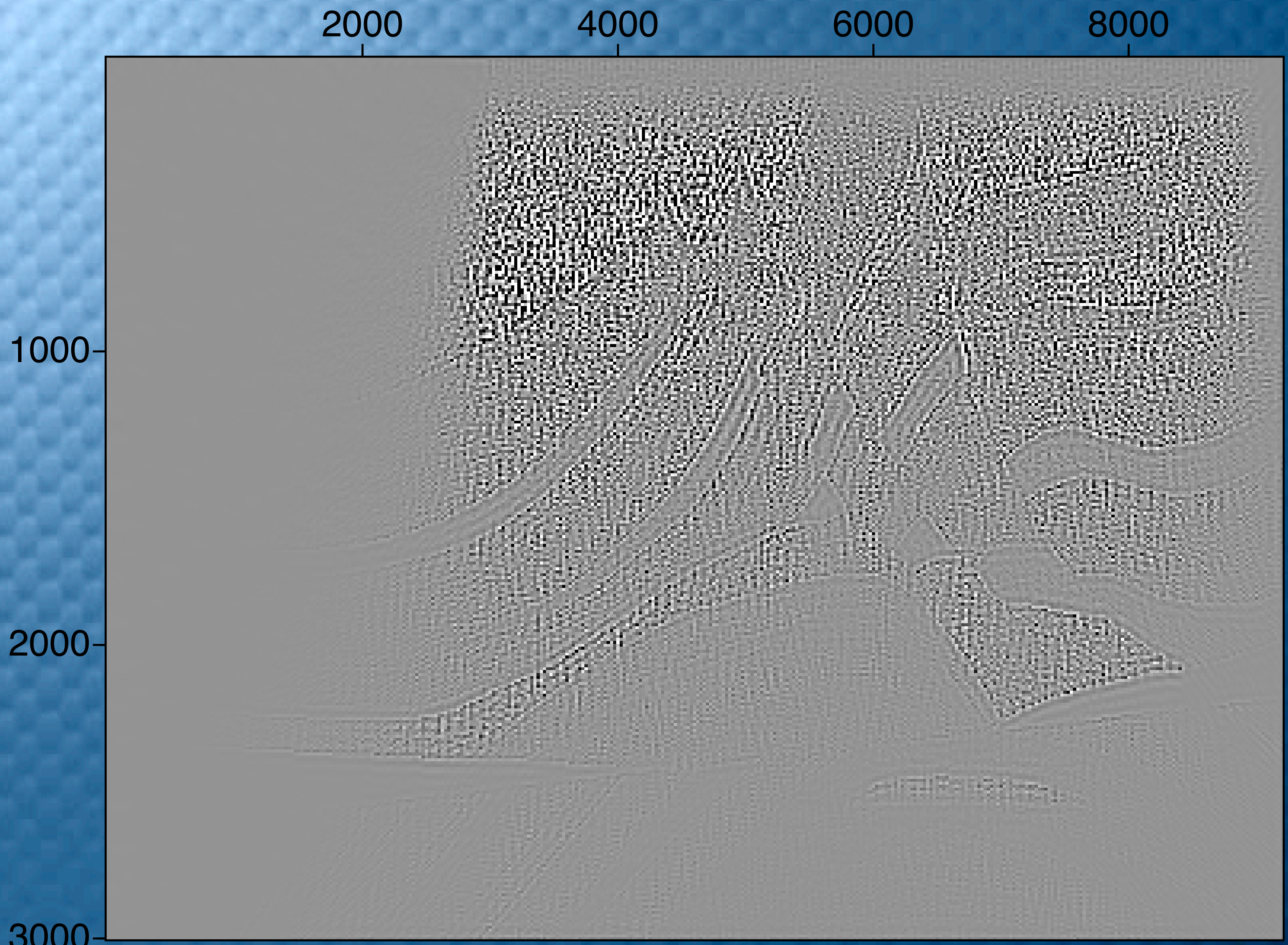
Preconditioned normal operator

Examples



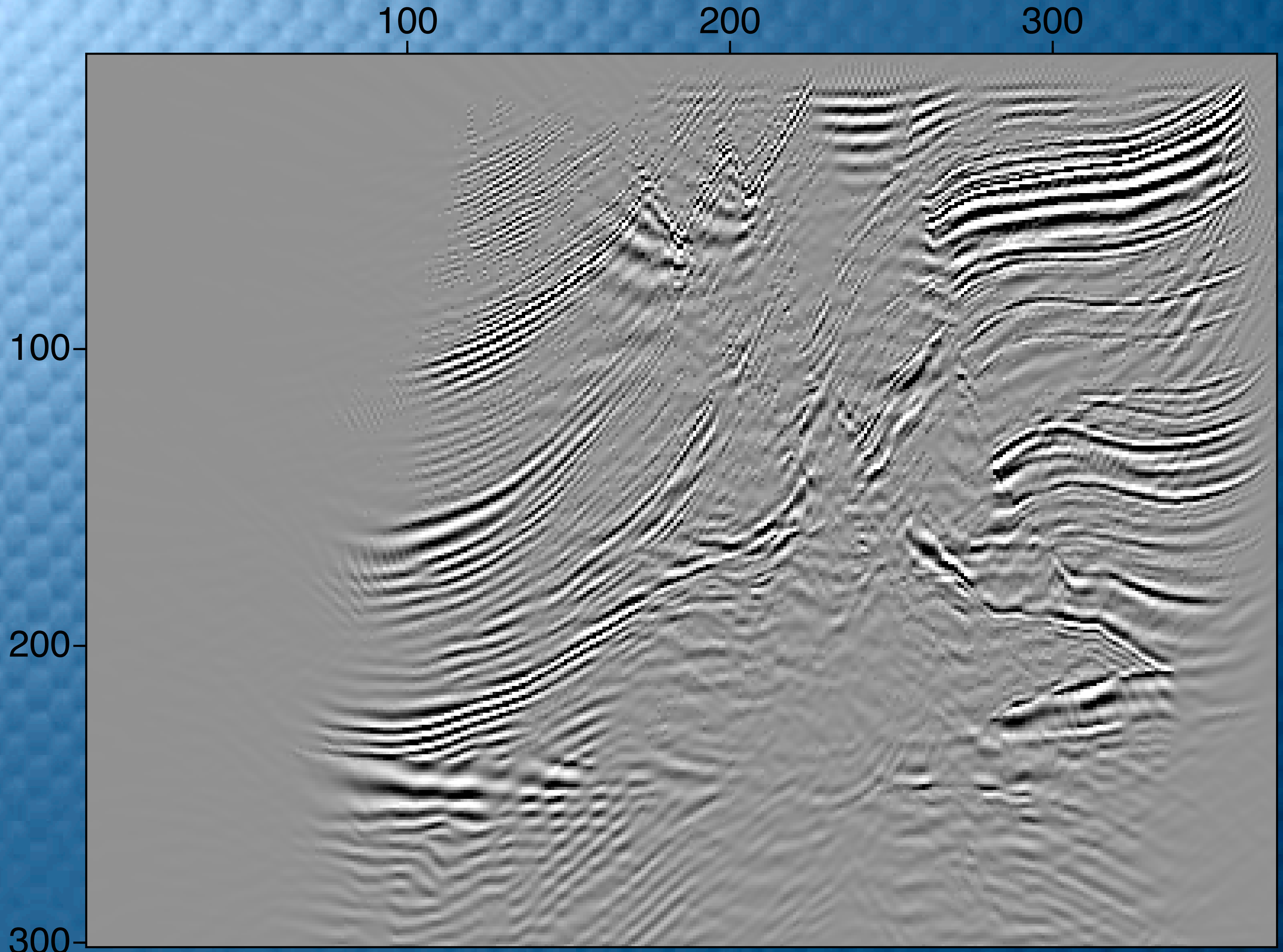
Noisy data SNR=0

Examples



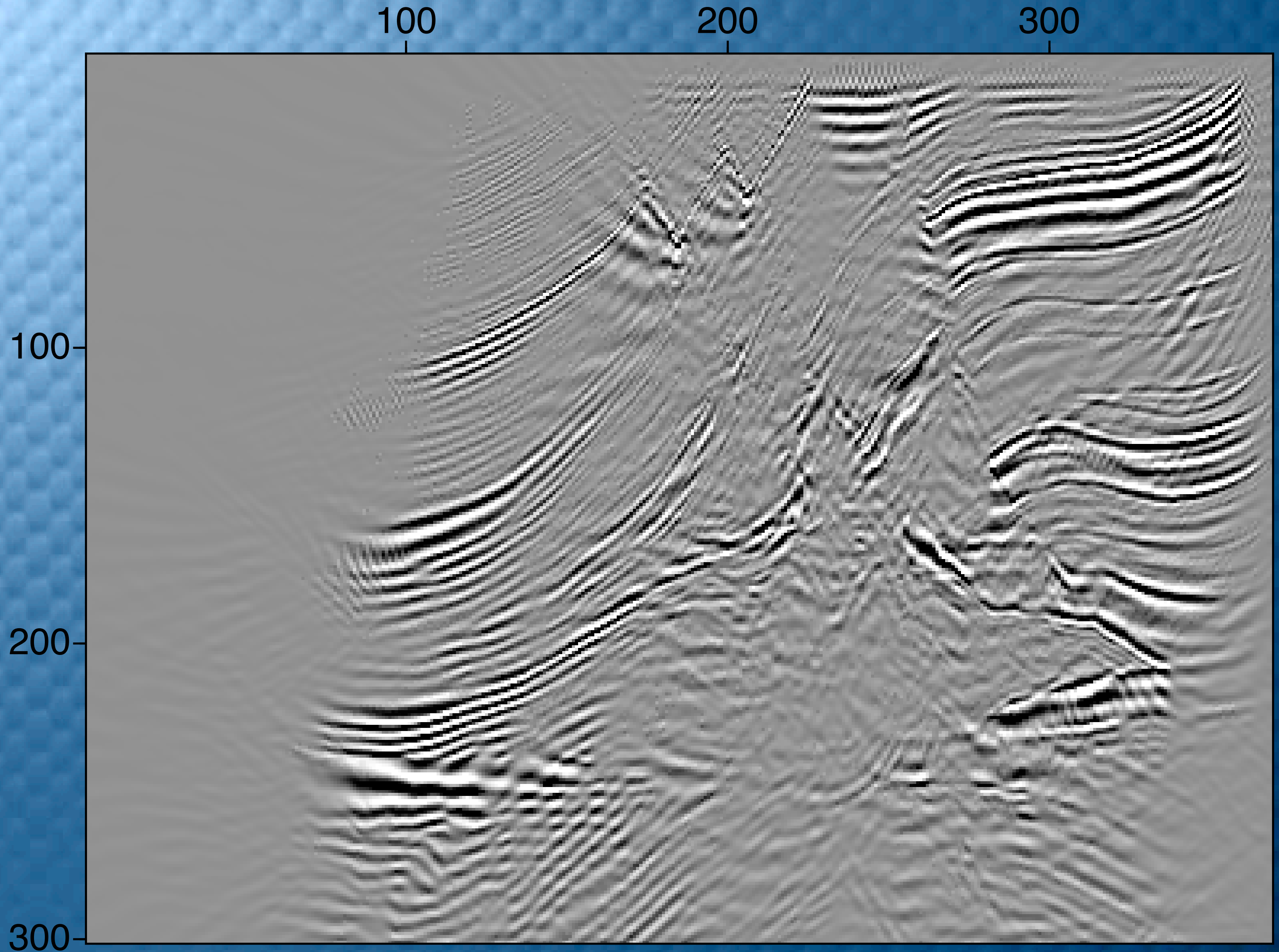
Inv. Curvelet Trans. Diag. Normal operator

Examples



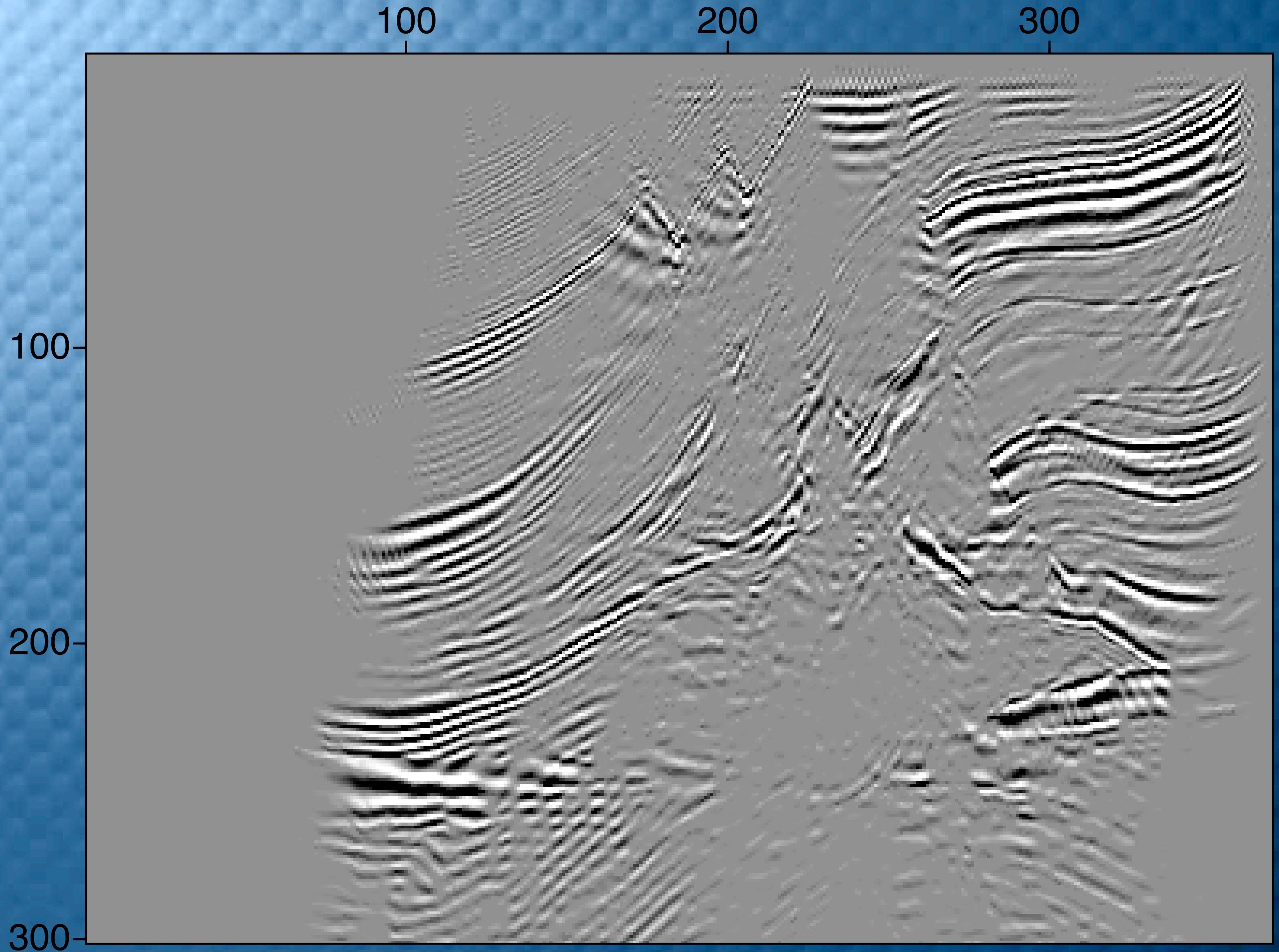
Thresholded

Examples



Thresholded and corrected

Examples



Optimized denoised

Conclusions

Aimed at compression of operators & model:

- ***Optimal*** **representation for** m
- ***Ultimate preconditioner***

Thresholding **brings us *close* to the solution**

- **Curvelets exploit *smoothness along* reflectors**
- **Constrained optimization is promising**
- **Finding appropriate *norm* is *crucial & open***

Improved the SNR!

Acknowledgements

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